Analogs of generalized resolvents of relations generated by pair of differential operator expressions one of which depends on spectral parameter in nonlinear manner

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Abstract. For the relations generated by pair of differential operator expressions one of which depends on the spectral parameter in the Nevanlinna manner we construct analogs of the generalized resolvents which are integro-differential operators.

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INTRODUCTION

We consider either on finite or infinite interval operator differential equation of arbitrary order

$$l_{\lambda}[y] = m[f], \ t \in \bar{\mathcal{I}}, \ \mathcal{I} = (a, b) \subseteq \mathbb{R}^1$$
(1)

in the space of vector-functions with values in the separable Hilbert space \mathcal{H} , where

$$l_{\lambda}[y] = l[y] - \lambda m[y] - n_{\lambda}[y], \qquad (2)$$

l[y], m[y] are symmetric operator differential expression. The order of $l_{\lambda}[y]$ is equal to r > 0. For the expression m[y] the subintegral quadratic form $m\{y, y\}$ of the Dirichlet integral $m[y, y] = \int_{\mathcal{I}} m\{y, y\} dt$ is nonnegative for $t \in \overline{\mathcal{I}}$. The leading coefficient of the expression m[y] may lack the inverse from $B(\mathcal{H})$ for any $t \in \overline{\mathcal{I}}$ and even it may vanish on some intervals. For the operator differential expression $n_{\lambda}[y]$ the form $n_{\lambda}\{y, y\}$ depends on λ in the Nevanlinna manner for $t \in \overline{\mathcal{I}}$. Therefore the order $s \geq 0$ of m[y] is even and $\leq r$.

In the Hilbert space $L^2_m(\mathcal{I})$ with metrics generated by the form m[y, y] for equation (1)-(2) we construct analogs $R(\lambda)$ of the generalized resolvents which in general are non-injective and which possess the following representation:

$$R(\lambda) = \int_{\mathbb{R}^1} \frac{dE_{\mu}}{\mu - \lambda} \tag{3}$$

where E_{μ} is a generalized spectral family for which E_{∞} is less or equal to the identity operator. (Abstract operators which possess such representation were studied in [16].)

This construction is based on a special reduction of the equation

$$l[y] = m[f] \tag{4}$$

to the first order system with weight. Here l and m are operator differential expressions which are not necessary symmetric (in contrast to (2)). For construction of $R(\lambda)$ we also introduce the characteristic operator of the equation

$$l_{\lambda}[y] = -\frac{(\Im l_{\lambda})[f]}{\Im \lambda}, \ t \in \bar{\mathcal{I}},$$
(5)

where $(\Im l_{\lambda})[f] = \frac{1}{2i}(l[f] - l^*[f])$. In the case r = 1, $n_{\lambda}[y] = H_{\lambda}(t)y$ (here the mentioned reduction is not needed) the resolvents $R(\lambda)$ was constructed in [21].

Further in the work we consider the boundary value problem obtained by adding to equation (1)-(2) the dissipative boundary conditions depending on a spectral parameter. We prove that for some boundary conditions solutions of such problems are generated by the operators $R(\lambda)$ if, in contrast to the case s = 0, $n_{\lambda}[y] = H_{\lambda}(t)y$, the boundary conditions contain the derivatives of vector-function f(t) that are taken on the ends of the interval.

In the case $n_{\lambda}[y] \equiv 0$ the results listed above are known [24], and $R(\lambda)$ is the generalized resolvent of the minimal relation generated by the pair of expressions l[y] and m[y]. For this case we show in the work that in the regular case all generalized resolvents are exhausted by the operators $R(\lambda)$, and thereby by virtue of [22] their full description with the help of boundary conditions is given. A review of other results for the case $n_{\lambda}[y] \equiv 0$ is in the work 23.

In the works [9], [10] the question of the conditions for holomorphy and continuous reversibility of the restrictions of maximal relations generated by $l_{\lambda}[y]$ (2) with $m[y] \equiv 0$, $n_{\lambda}[y] = H_{\lambda}(t)y$ in $L^2_{\Im H_{\lambda_0}(t)/\Im \lambda_0}$ ($\Im \lambda_0 \neq 0$) and also by the integral equation with the Nevanlinna matrix measure was studied (using some of the results from [22]). We remark that the relations inverse to those ones considered in [9], [10] do not possess the representation (3). Also we note that the resolvent equation (1)-(2) is not reduced to the equations considered in [9], [10].

Many questions, that concern differential operators and relations in the space of vectorfunctions, are considered in the monographs [2, 4, 5, 18, 27, 28, 34, 35] containing an extensive literature. The method of studying of these operators and relations based on use of the abstract Weyl function and its generalization (Weyl family) was proposed in [14, 12, 13].

A preliminary version of results of this paper is contained in preprint [25]. The expansion formulae in the solutions of the homogeneous equation (1) will be obtained in our next paper.

We denote by (.) and $\|\cdot\|$ the scalar product and the norm in various spaces with special indices if it is necessary. For differential expression L we denote $\Re l = \frac{1}{2}(l+l^*), \Im l = \frac{1}{2i}(l-l^*).$

Let an interval $\Delta \subseteq \mathbb{R}^1$, f(t) $(t \in \Delta)$ be a function with values in some Banach space B. The notation $f(t) \in C^k(\Delta, B)$, k = 0, 1, ... (we omit the index k if k = 0) means, that in any point of $\Delta f(t)$ has continuous in the norm $\|\cdot\|_{B}$ derivatives of order up to and including l that are taken in the norm $\|\cdot\|_{B}$; if Δ is either semi-open or closed interval then on its ends belonging to Δ the one-side continuous derivatives exist. The notation $f(t) \in C_0^k(\Delta, B)$ means that $f(t) \in C^k(\Delta, B)$ and f(t) = 0 in the neighbourhoods of the ends of Δ .

1. The reduction of equation (4) to the first order system of canonical type WITH WEIGHT. THE GREEN FORMULA

We consider in the separable Hilbert space \mathcal{H} equation (4), where l[y] and m[f] are differential expressions (that are not necessary symmetric) with sufficiently smooth coefficients from $B(\mathcal{H})$ and of orders r > 0 and s correspondingly. Here $r \ge s \ge 0$, s is even and these expressions are presented in the divergent form. Namely:

$$l[y] = \sum_{k=0}^{r} i^{k} l_{k}[y], \qquad (6)$$

where $l_{2j} = D^j p_j(t) D^j$, $l_{2j-1} = \frac{1}{2} D^{j-1} \{ Dq_j(t) + s_j(t) D \} D^{j-1}$, $p_j(t), q_j(t), s_j(t) \in C^j(\overline{\mathcal{I}}, B(\mathcal{H})), D = d/dt; m[f] \text{ is defined in a similar way with } s \text{ instead of } r \text{ and } \tilde{p}_j(t), \ \tilde{q}_j(t), \ \tilde{s}_j(t) \in B(\mathcal{H}) \text{ instead of } p_j(t), \ q_j(t), \ s_j(t) \in D(\mathcal{H}) \}$

In the case of even $r = 2n \ge s, p_n^{-1} \in B(\mathcal{H})$ we denote

$$Q(t,l) = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} = \frac{J}{i}, \quad S(t,l) = Q(t,l), \quad (7)$$

$$H(t, l) = \|h_{\alpha\beta}\|_{\alpha, \beta=1}^{2}, \ h_{\alpha\beta} \in B(\mathcal{H}^{n}),$$
(8)

where I_n is the identity operator in $B(\mathcal{H}^n)$; h_{11} is a three diagonal operator matrix whose elements under the main diagonal are equal to $(\frac{i}{2}q_1, \ldots, \frac{i}{2}q_{n-1})$, the elements over the main diagonal are equal to $(-\frac{i}{2}s_1, \ldots, -\frac{i}{2}s_{n-1})$, the elements on the main diagonal are equal to $(-p_0, \ldots, -p_{n-2}, \frac{1}{4}s_np_n^{-1}q_n - p_{n-1})$; h_{12} is an operator matrix with the identity operators I_1 under the main diagonal, the elements on the main diagonal are equal to $(0, \ldots, 0, -\frac{i}{2}s_np_n^{-1})$, the rest elements are equal to zero; h_{21} is an operator matrix with identity operators I_1 over the main diagonal, the elements on the main diagonal are equal to $(0, \ldots, 0, \frac{i}{2}p_n^{-1}q_n)$, the rest elements are equal to zero; $h_{22} = \text{diag}(0, \ldots, 0, p_n^{-1})$.

Also in this case we denote 1

$$W(t, l, m) = C^{*-1}(t, l) \left\{ \|m_{\alpha\beta}\|_{\alpha, \beta=1}^{2} \right\} C^{-1}(t, l), m_{\alpha\beta} \in B(\mathcal{H}^{n}),$$
(9)

where m_{11} is a tree diagonal operator matrix whose elements under the main diagonal are equal to $\left(-\frac{i}{2}\tilde{q}_1, \ldots, -\frac{i}{2}\tilde{q}_{n-1}\right)$, the elements over the main diagonal are equal to $\left(\frac{i}{2}\tilde{s}_1, \ldots, \frac{i}{2}\tilde{s}_{n-1}\right)$, the elements on the main diagonal are equal to $(\tilde{p}_0, \ldots, \tilde{p}_{n-1})$; $m_{12} = \text{diag}\left(0, \ldots, 0, \frac{i}{2}\tilde{s}_n\right)$, $m_{21} = \text{diag}\left(0, \ldots, 0, -\frac{i}{2}\tilde{q}_n\right), m_{22} = \text{diag}\left(0, \ldots, 0, \tilde{p}_n\right)$.

Operator matrix $C(t, \bar{l})$ is defined by the condition

$$C(t,l) \operatorname{col} \left\{ f(t), f'(t), \dots, f^{(n-1)}(t), f^{(2n-1)}(t), \dots, f^{(n)}(t) \right\} = \operatorname{col} \left\{ f^{[0]}(t|l), f^{[1]}(t|l), \dots, f^{[n-1]}(t|l), f^{[2n-1]}(t|l), \dots, f^{[n]}(t|l) \right\},$$
(10)

where $f^{[k]}(t|L)$ are quasi-derivatives of vector-function f(t) that correspond to differential expression L.

The quasi-derivatives corresponding to l are equal (cf. [33]) to

$$y^{[j]}(t|l) = y^{(j)}(t), \quad j = 0, \dots, \quad \left[\frac{r}{2}\right] - 1, \tag{11}$$

$$y^{[n]}(t|l) = \begin{cases} p_n y^{(n)} - \frac{i}{2} q_n y^{(n-1)}, \ r = 2n \\ -\frac{i}{2} q_{n+1} y^{(n)}, \ r = 2n+1 \end{cases},$$
(12)

$$y^{[r-j]}(t|l) = -Dy^{[r-j-1]}(t|l) + p_j y^{(j)} + \frac{i}{2} \left[s_{j+1} y^{(j+1)} - q_j y^{(j-1)} \right], \ j = 0, \ \dots, \ \left[\frac{r-1}{2} \right], \ q_0 \equiv 0$$
(13)

 $^{{}^{1}}W(t,l,m)$ is given for the case s = 2n. If s < 2n one have set the corresponding elements of operator matrices $m_{\alpha\beta}$ be equal to zero. In particular if s < 2n then $m_{12} = m_{21} = m_{22} = 0$ and therefore $W(t,l,m) = \text{diag}(m_{11},0)$ in view of (14).

At that $l[y] = y^{[r]}(t|l)$. The quasi-derivatives $y^{[k]}(t|m)$ corresponding to m are defined in the same way with even s instead of r and $\tilde{p}_j, \tilde{q}_j, \tilde{s}_j$ instead of p_j, q_j, s_j .

It is easy to see that

$$C(t,l) = \begin{pmatrix} I_n & 0\\ C_{21} & C_{22} \end{pmatrix}, \quad C_{\alpha\beta} \in B(\mathcal{H}^n),$$
(14)

 C_{21}, C_{22} are upper triangular operator matrices with diagonal elements $\left(-\frac{i}{2}q_1, \ldots, -\frac{i}{2}q_n\right)$ and $\left((-1)^{n-1}p_n, (-1)^{n-2}p_n, \ldots, p_n\right)$ correspondingly.

In the case of odd r = 2n + 1 > s we denote

$$Q(t,l) = \begin{cases} J/i \oplus q_{n+1} \\ q_1 \end{cases}, \quad S(t,l) = \begin{cases} J/i \oplus s_{n+1}, & n > 0 \\ s_1, & n = 0 \end{cases},$$
(15)

$$H(t, l) = \begin{cases} \|h_{\alpha\beta}\|_{\alpha,\beta=1}^{2}, & n > 0\\ p_{0}, & n = 0 \end{cases},$$
(16)

where $B(\mathcal{H}^n) \ni h_{11}$ is a three-diagonal operator matrix whose elements under the main diagonal are equal to $(\frac{i}{2}q_1, \ldots, \frac{i}{2}q_{n-1})$, the elements over the main diagonal are equal to $(-\frac{i}{2}s_1, \ldots, -\frac{i}{2}s_{n-1})$, the elements on the main diagonal are equal to $(-p_0, \ldots, -p_{n-1})$, the rest elements are equal to zero. $B(\mathcal{H}^{n+1}, \mathcal{H}^n) \ni h_{12}$ is an operator matrix whose elements with numbers j, j - 1 are equal to $I_1, j = 2, \ldots, n$, the element with number n, n + 1is equal to $\frac{1}{2}s_n$, the rest elements are equal to zero. $B(\mathcal{H}^n, \mathcal{H}^{n+1}) \ni h_{21}$ is an operator matrix whose elements with numbers j - 1, j are equal to $I_1, j = 2, \ldots, n$, the element with number n + 1, n is equal to $\frac{1}{2}q_n$, the rest elements are equal to zero. $B(\mathcal{H}^{n+1}) \ni h_{22}$ is an operator matrix whose last row is equal to $(0, \ldots, 0, -iI_1, -p_n)$, last column is equal to $col(0, \ldots, 0, iI_1, -p_n)$, the rest elements are equal to zero.

Also in this case we denote 2

$$W(t, l, m) = \|m_{\alpha\beta}\|_{\alpha,\beta=1}^{2}, \qquad (17)$$

where m_{11} is defined in the same way as m_{11} (9). $B\left(\mathcal{H}^{n+1}, \mathcal{H}^n\right) \ni m_{12}$ is an operator matrix whose element with number n, n+1 is equal to $-\frac{1}{2}\tilde{s}_n$, the rest elements are equal to zero. $B\left(\mathcal{H}^n, \mathcal{H}^{n+1}\right) \ni m_{21}$ is an operator matrix whose element with number n+1, n is equal to $-\frac{1}{2}\tilde{q}_n$, the rest elements are equal to zero. $B\left(\mathcal{H}^{n+1}\right) \ni m_{22} = \text{diag}\left(0, \ldots, 0, \tilde{p}_n\right)$.

Obviously for H(t, l) (8), (16) and W(t, l, m) (9), (17) one has

$$H^{*}(t,l) = H(t,l^{*}), W^{*}(t,l,m) = W(t,l,m^{*}).$$
(18)

Lemma 1.1. Let the order of $\Im l$ is even. Then

$$\Im H(t,l) = W(t,l,-\Im l) = W(t,l^*,-\Im l).$$
 (19)

Proof. Let us prove the first equality in (19) for even r = 2n. Let us represent H(t, l) (8) in the form

$$H(t, l) = A(t, l) + B(t, l),$$
(20)

 $^{^{2}}$ See the previous footnote

where A(t, l) = H(t, l) - B(t, l) and

$$B(t,l) = \|B_{jk}\|_{j,k=1}^{2}, \ B_{jk} \in B(\mathcal{H}^{n}),$$
(21)

$$B_{11} = \operatorname{diag}\left(0, ..., 0, s_n p_n^{-1} q_n/4\right), \quad B_{12} = \operatorname{diag}\left(0, ..., 0, -i s_n p_n^{-1}/2\right), \tag{22}$$

$$B_{21} = \operatorname{diag}\left(0, ..., 0, ip_n^{-1}q_n/2\right), \quad B_{22} = \operatorname{diag}\left(0, ..., 0, p_n^{-1}\right).$$
(23)

In view of (14), (21) - (23) one has

$$B(t,l) C(t,l) = \|u_{jk}\|_{j,k=1}^{2n}, \ u_{jk} \in B(\mathcal{H}),$$
(24)

 $u_{n2n} = -is_n/2, \ u_{2n \ 2n} = I_1, \text{ rest } u_{jk} = 0.$ Hence

$$C^{*}(t,l) B(t,l) C(t,l) = ||v_{jk}||_{j,k=1}^{2n}, v_{jk} \in B(\mathcal{H}),$$

 $v_{n \ 2n} = -\frac{i}{2} (s_n - q_n^*), v_{2n \ 2n} = p_n^*, \text{ rest } v_{jk} = 0.$ Hence $C^*(t,l) \Im H(t,l) C(t,l) = C^*(t,l) W(t,l,-\Im l) C(t,l)$ in view of (8), (9), (10), (20) and the divergent form of the expression $-\Im l$ that follows from (6). The first equality in (19) for even r is proved. Its proof for odd r follows from (16), (17).

One can see from the proof that

$$W(t, l, \Im l) = -\Im H(t, l).$$
⁽²⁵⁾

The second equality in (19) is a corollary of (25) and (18). Lemma 1.1 is proved

For sufficiently smooth vector-function f(t) by corresponding capital letter we denote (if f(t) has a subscript then we add the same subscript to F)

$$\mathcal{H}^{r} \ni F(t, l, m) = \begin{cases} \left(\sum_{j=0}^{s/2} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \dots \oplus 0, \quad r = 2n, \qquad r = 2n+1 > 1, s < 2n, \\ \left(\sum_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \dots \oplus 0 \oplus \left(-if^{(n)}(t)\right), \quad r = 2n+1 > 1, s = 2n, \\ f(t), \qquad r = 1, \\ \left(\sum_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus f^{[r-j]}(t|l)\right), \qquad r = s = 2n \end{cases}$$
(26)

From now on in equation (4)

$$p_n^{-1}(t) \in B(\mathcal{H}) \ (r=2n); \ (q_{n+1}(t)+s_{n+1}(t))^{-1} \in B(\mathcal{H}) \ (r=2n+1).$$

Theorem 1.1. Equation (4) is equivalent to the following first order system

$$\frac{i}{2}\left(\left(Q\left(t\right)\vec{y}\right)' + S\left(t\right)\vec{y}'\right) + H\left(t\right)\vec{y} = W\left(t\right)F\left(t\right)$$
(27)

with coefficients Q(t) = Q(t,l), S(t) = S(t,l) (7), (15), H(t) = H(t,l) (8), (16), weight $W(t) = W(t, l^*, m)$, and with $F(t) = F(t, l^*, m)$ that are obtained from (9), (17) and (26)

correspondingly with l^* instead of l. Namely if y(t) is a solution of equation (4) then

$$\vec{y}(t) = \vec{y}(t, l, m, f) = \begin{cases} \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus \left(y^{[r-j]}(t \mid l) - f^{[s-j]}(t \mid m)\right)\right), & r = 2n \\ \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus \left(y^{[r-j]}(t \mid l) - f^{[s-j]}(t \mid m)\right)\right) \oplus \left(-iy^{(n)}(t)\right), & r = 2n + 1 > 1 \\ \text{(here } f^{[k]}(t \mid m) \equiv 0 \text{ as } k < \frac{s}{2} \right) \\ y(t), & r = 1 \end{cases}$$
(28)

is a solution of (27) with the coefficients, weight and F(t) menfioned above. Any solution of equation (27) with such coefficients, weight and F(t) is equal to (28), where y(t) is some solution of equation (4).

Let us notice that different vector-functions f(t) can generate different right-hand-sides of equation (27) but unique right-hand-side of equation (4).

Proof. We need the following three lemmas.

Lemma 1.2. Let L_{α} be a differential expression of l type and of order α . Let us add to L_{α} the expressions of $i^k l_k$ type, where $k = \alpha + 1, \ldots, \beta$, with coefficients equal to zero. We obtain the expressions L_{β} which formally has the order β , but in fact L_{β} and L_{α} coincide. Then for sufficiently smooth vector-function f(t)

$$f^{\left[\beta-j\right]}\left(t\left|L_{\beta}\right.\right) = \begin{cases} f^{\left[\alpha-j\right]}\left(t\left|L_{\alpha}\right.\right), & j = 0, \dots, \left[\frac{a+1}{2}\right], \\ 0, & j = \left[\frac{a+1}{2}\right] + 1, \dots, \left[\frac{\beta}{2}\right] \end{cases}$$

(here $f^{[0]}(t|L_1)$ is defined by (12) with r = 1).

Proof. Proof of Lemma 1.2 follows from formulae (12) - (13) for quasi-derivatives.

Lemma 1.3. Let $f(t) \in C^s([\alpha, \beta], \mathcal{H}), y(t)$ is a solution of corresponding equation (4). Then the sequence $f_k(t) \in C^{\infty}([\alpha, \beta], \mathcal{H})$ and solutions $y_k(t)$ of equation (4) with $f(t) = f_k(t)$ exist such that

$$f_k(t) \xrightarrow{C^s([\alpha,\beta],\mathcal{H})} f(t), \ y_k(t) \xrightarrow{C^r([\alpha,\beta],\mathcal{H})} y(t).$$

This is trivial consequence of Weierstrass theorem for vector-functions [36] and formula (1.21) from [11].

Lemma 1.4. Let vector-function $f(t) \in C^s(\overline{\mathcal{I}}, \mathcal{H})$. Then

$$W(t, l^{*}, m) F(t, l^{*}, m) = \begin{cases} \begin{pmatrix} s/2^{-1} \oplus \left(f^{[s-j]}(t \mid m) + \left(f^{[s-j-1]}(t \mid m) \right)' \right) \end{pmatrix} \oplus \\ \oplus f^{[s/2]}(t \mid m) \oplus 0 \oplus \dots \oplus 0, \\ \begin{pmatrix} s/2^{-1} \oplus \left(f^{[s-j]}(t \mid m) + \left(f^{[s-j-1]}(t \mid m) \right)' \right) \end{pmatrix} \oplus \\ \oplus 0 \oplus \dots \oplus 0 \oplus \left(-if^{[n]}(t \mid m) \right), \\ \tilde{p}_{0}(t) f(t) \oplus 0 \oplus \dots \oplus 0, \\ \begin{bmatrix} \begin{pmatrix} s/2^{-1} \oplus \left(f^{[s-j]}(t \mid m) + \left(f^{[s-j-1]}(t \mid m) \right)' \right) \\ \end{pmatrix} \oplus \\ \oplus 0 \oplus \dots \oplus 0 \end{bmatrix} + H(t, l) \left(0 \oplus \dots \oplus 0 \oplus f^{[n]}(t \mid m) \right), \\ r = s = 2n \end{cases}$$

$$(29)$$

Let us notice that $W(t, l^*, m)F(t, l^*, m)$ does not change if the null-components in $F(t, l^*, m)$ we change by any \mathcal{H} -valued vector-functions.

Proof. Let us prove Lemma 1.4 for r = s = 2n. It is sufficient to verify that

$$\left(\|m_{\alpha\beta}(t)\|_{\alpha,\beta=1}^{2} \right) col \left\{ f(t), f'(t), ..., f^{(n-1)}(t), f^{(2n-1)}(t), ..., f^{(n)}(t) \right\} = = C^{*}(t, l^{*}) \left\{ \left[\left(\sum_{j=0}^{n-1} \left(f^{[r-j]}(t \mid m) + \left(f^{[r-j-1]}(t \mid m) \right)' \right) \right) \oplus 0 \oplus ... \oplus 0 \right] + + H(t \mid l) \left(0 \oplus 0 \oplus ... \oplus 0 \oplus f^{[n]}(t \mid m) \right) \right\}.$$
(30)

But in view of (9), (12), (13) the left side of equality (30) is equal to

$$\left(\sum_{j=0}^{n-1} \oplus \left(f^{[r-j]}(t \mid m)\right) + \left(f^{[r-j-1]}(t \mid m)\right)'\right) \oplus O \oplus ... \oplus O \oplus f^{[n]}(t \mid m).$$

And hence equality (30) is true since $C(t, l^*)[\ldots] = [\ldots]$ and the last column of $C^*(t, l^*) H(t, l)$ is equal to *col* $(0, \ldots, 0, I_1)$ in view of (8), (14).

The proof for r = 2n + 1, s = 2n is carried out via direct calculation using (17), (12), (13).

The proof for s < 2n follows from the case s = 2n considered above, Lemmas 1.2, 1.3 and fact that elements $u_{jk} \in B(\mathcal{H})$ of matrix $W(t, l^*, m)$ are equal to zero if s < 2n and i > s/2 or j > s/2. Lemma 1.4 is proved.

Let us return to the proof of Theorem 1.1. Let y(t) is a solution of equation (4). Then

$$\frac{i}{2}\left\{ \left(Q\left(t,l\right)\vec{y}\left(t,l,m,0\right)\right)' + S\left(t,l\right)\vec{y}'\left(t,l,m,0\right) \right\} - H\left(t,l\right)\vec{y}\left(t,l,m,0\right) = \operatorname{diag}\left(y^{[r]}\left(t\,|l\right),0,\ldots,0\right) \quad (31)$$

in view of formulae that are analogues to formulae (4.10), (4.11), (4.24), (4.25) from [26]. Using (31) and Lemma 1.4 we show via direct calculations that $\vec{y}(t, l, m, f)$ (28) is a solution

of (27) for r = s = 2n, r = 2n + 1, s = 2n. Therefore in view of Lemmas 1.2, 1.3 $\vec{y}(t, l, m, f)$ is a solution of (27) for s < 2n.

Conversely let $\vec{y}(t) = col(y_1, \ldots, y_r)$ is a solution of (27). Let y(t) is a solution of Cauchy problem that is obtained by adding the initial condition $\vec{y}(0, l, m, f) = \vec{y}(0)$ to equation (4). Then $\vec{y}(t) = \vec{y}(t, l, m, f)$ in view of existence and uniqueness theorem. Theorem 1.1 is proved

Let us notice that Theorem 1.1 remains valid if null-components of $F(t, l^*, m)$ we change by any \mathcal{H} -valued vector-functions.

For differential expression
$$L = \sum_{k=0}^{n} i^k L_k$$
, where $L_{2j} = D^j P_j(t) D^j$,
 $L_{2j-1} = \frac{1}{2} D^{j-1} \{ DQ_j(t) + S_j(t) \ D \} D^{j-1}$, we denote by
 $L[f, g] = \int_{\mathcal{T}} L\{f, g\} dt$, (32)

the bilinear form which corresponds to Dirichlet integral for this expression. Here

$$L\{f,g\} = \sum_{j=0}^{[R/2]} \left(P_j(t) f^{(j)}(t), g^{(j)}(t) \right) + \frac{i}{2} \sum_{j=1}^{[\frac{R+1}{2}]} \left(S_j(t) f^{(j)}(t), g^{(j-1)}(t) \right) - \left(Q_j(t) f^{(j-1)}(t), g^{(j)}(t) \right)$$
(33)

Theorem 1.2 (On the relationships between bilinear forms). Let f(t), y(t), $f_k(t)$, $y_k(t)$ (k = 1, 2) be sufficiently smooth vector-function. Starting from these functions by the formulae (26), (28) we construct F(t, l, m), $F_k(t, l, m)$, $\vec{y}(t, l, m, f)$, $\vec{y}_k(t, l, m, f_k)$. Then: 1.

$$(W(t, l, m) F_1(t, l, m), F_2(t, l, m)) = m \{f_1, f_2\}.$$
(34)

2. a) If the order of $\Im l$ is even, then

$$(W(t,l,-\Im l)\,\vec{y}\,(t,l,m,f)\,,\,\vec{y}\,(t,\,l,\,m,\,f)) - \Im\,(W(t,\,l^*,m^*)\,\vec{y}\,(t,\,l,m,f))\,,\,F(t,l^*,m) = \\ = -\,(\Im l\,\{y,y\} - \Im\,(m^*\,\{y,f\})\,.$$
(35)

b)

$$m \{y_1, f_2\} - m \{f_1, y_2\} = = (W(t, l, m) \vec{y}_1(t, l, m, f_1), F_2(t, l, m)) - (W(t, l^*, m) F_1(t, l^*, m), \vec{y}_2(t, l^*, m^*, f_2)), \quad (36)$$

although for r = s the corresponding terms in the right-and left-hand side of (35) and (36) do not coincide.

Proof. 1. follows from (9), (17), (26), (33).

2. Let r = s = 2n. For convenience when using notations of (26) type we omit the argument m. For example by $F(t, l^*)$ we denote $F(t, l^*, m)$.

a) We denote

$$\mathcal{F}(t,m) = col\left\{0,...,0,f^{[2n-1]}(t|m),...,f^{[n]}(t|m)\right\} \in \mathcal{H}^{r}.$$
(37)

One has

$$(W(t,l,-\Im l) \vec{y}(t,l,m,f), \vec{y}(t,l,m,f)) = (W(t,l,-\Im l) Y(t,l), Y(t,l)) - - (W(t,l,-\Im l) Y(t,l), \mathcal{F}(t,m)) - (W(t,l,-\Im l) \mathcal{F}(t,m), Y(t,l)) + + (W(t,l,-\Im l) \mathcal{F}(t,m), \mathcal{F}(t,m)) = - (\Im l) [y,y] + + 2\Re \left(p_r^{*-1} y^{[n]}(t|\Im l), f^{[n]}(t|m) \right) + \Im \left(p_r^{-1} f^{[n]}(t|m), f^{[n]}(t|m) \right).$$
(38)

Here the last equality follows from (18), (34), (29), (8). On the other hand we have

$$\begin{split} \Im\left(W\left(t,l^{*},m^{*}\right)\vec{y}\left(t,l,m,f\right),F\left(t,l^{*}\right)\right) &= \Im\left(W\left(t,l^{*},m^{*}\right)Y\left(t,l^{*}\right),F\left(t,l^{*}\right)\right) + \\ &+ \Im\left(W\left(t,l^{*},m^{*}\right)\left(\left(Y\left(t,l\right)-Y\left(t,l^{*}\right)\right)-\mathcal{F}\left(t,m\right)\right),F\left(t,l^{*}\right)\right) = \Im\left(m^{*}\left\{y,f\right\}\right) + \\ &+ 2\Re\left(p_{r}^{*-1}y^{[n]}\left(t|\Im l\right),f^{[n]}\left(t|m\right)\right) + \Im\left(p_{n}^{-1}f^{[n]}\left(t|m\right),f^{[n]}\left(t|m\right)\right). \end{split}$$
(39)

Here the last equality is proved similarly to (38) taking into account that $y^{[n]}(t|l) - y^{[n]}(t|l^*) = 2iy^{[n]}(t|\Im l)$. Comparing (38), (39) we obtain (35).

b) In view of (28), (34), (18) and Lemma 1.4 we have

$$(W(t,l,m) \vec{y}_{1}(t,l,m,f_{1}), F_{2}(t,l)) = m \{y_{1}(t,l,m,f_{1}), f_{2}\} - \left(\mathcal{F}_{1}(t,m), H(t,l^{*}) \operatorname{col} \{0, ..., 0, f_{2}^{[n]}(t \mid m^{*})\}\right) = m \{y_{1}, f_{2}\} - \left(p_{n}^{-1}f_{1}^{[n]}(t \mid m), f_{2}^{[n]}(t \mid m^{*})\right).$$
(40)

Similarly

$$(W(t,l^*,m)F_1(t,l^*),\vec{y}_2(t,l^*,m^*,f_2)) = m\{f_1,y_2\} - \left(p_n^{-1}f_1^{[n]}(t\,|m),f_2^{[n]}(t\,|m^*)\right)$$
(41)

Comparing (40), (41) we obtain (36).

For r = 2n + 1, s = 2n or $r = 2n + 1 \lor 2n$, s < 2n, the corresponding terms in (35), (36) coincide in view of (9), (17), (26), (28), (34). For example in these cases

$$(W(t, l, -\Im l) \, \vec{y} \, (t, l, m, f) \, , \vec{y} \, (t, l, m, f)) = ((W(t, l, -\Im l)) \, Y(t, l) \, , Y(t, l)) = - \, (\Im l) \, \{y, y\}$$

Theorem 1.2 is proved.

Let us notice that Theorem 1.2 remains valid if null-components in $F_k(t, l, m)$, $F(t, l^*, m)$, $F_1(t, l^*, m)$ we change by any \mathcal{H} -valued vector-functions.

Theorem 1.3 (The Green formula). Let l_k , m_k (k = 1, 2) are differential expressions of l(6), m type correspondingly. The orders of l_k are equal to r, the orders m_k are different in general, even and are equal to $s_k \leq r$. Let $y_k(t) \in C^r([\alpha, \beta], \mathcal{H}), f_k(t) \in C^{s_k}([\alpha, \beta], \mathcal{H}),$ and $l_k[y_k] = m_k[f_k], k = 1, 2$. Then

$$\int_{\alpha}^{\beta} m_1 \{f_1, y_2\} dt - \int_{\alpha}^{\beta} m_2^* \{y_1, f_2\} dt - \int_{\alpha}^{\beta} (l_1 - l_2^*) \{y_1, y_2\} dt = \\ = \left(\frac{i}{2} \left(Q(t, l_1) + Q^*(t, l_2)\right) \vec{y}_1(t, l_1, m_1, f_1), \vec{y}_2(t, l_2, m_2, f_2)\right) \Big|_{\alpha}^{\beta}, \quad (42)$$

where $Q(t, l_k)$, $\vec{y}_k(t, l_k, m_k, f_k)$ correspond to equations $l_k[y] = m_k[f]$ by formulae (7), (15), (28) with l_k, m_k, y_k, f_k instead of l, m, y, f correspondingly.

Proof. We need the following

Lemma 1.5. For sufficiently smooth vector-function $g_1(t)$, $g_2(t)$ one has

$$((H(t, l_1) - H(t, l_2^*)) \vec{g}_1(t, l_1, m_1, 0), \vec{g}_2(t, l_2, m_2, 0)) = \begin{cases} -(l_1 - l_2^*) \{g_1, g_2\}, & r = 2n \\ -(l_1 - l_2^*) \{g_1, g_2\} + (l_{2n+1}^1 - l_{2n+1}^{2^*}) \{g_1, g_2\}, & r = 2n + 1, \end{cases}$$
(43)

where l_{2n+1}^k are the analogs of l_{2n+1} .

Proof. Let r = 2n. Then in view of (20)-(24), (28), (10), (18) we have

$$\begin{split} \left(\left(H\left(t, l_{1}\right) - H\left(t, l_{2}^{*}\right) \right) \vec{g}_{1}\left(t, l_{1}, m_{1}, 0\right), \vec{g}_{2}\left(t, l_{2}, m_{2}, 0\right) \right) = \\ &= \left(\left(A\left(t, l_{1}\right) - A\left(t, l_{2}^{*}\right) \right) \vec{g}_{1}\left(t, l_{1}, m_{1}, 0\right), \vec{g}_{2}\left(t, l_{2}, m_{2}, 0\right) \right) + \\ &+ \left(C^{*}\left(t, l_{2}\right) B\left(t, l_{1}\right) C\left(t, l_{1}\right) col \left\{ g_{1}, g_{1}', \dots, g_{1}^{(n-1)}, g_{1}^{(2n-1)}, \dots, g_{1}^{(n)} \right\}, \\ &\quad col \left\{ g_{2}, g_{2}', \dots, g_{2}^{(n-1)}, g_{2}^{(2n-1)}, \dots, g_{2}^{(n)} \right\} \right) - \\ &- \left(col \left\{ g_{1}, g_{1}', \dots, g_{1}^{(2n-1)}, \dots, g_{1}^{(n)} \right\}, C^{*}\left(t, l_{1}\right) B\left(t, l_{2}\right) C\left(t, l_{2}\right) col \left\{ g_{2}, g_{2}', \dots, g_{2}^{(n-1)}, g_{2}^{(2n-1)}, \dots, g_{2}^{(n)} \right\} \right) = \\ &= - \left(l_{1} - l_{2}^{*} \right) \left\{ f, g \right\}. \end{split}$$

The proof of (43) for r = 2n + 1 follows directly from (16), (28). Lemma 1.5 is proved. \Box

Now Green formula (42) is obtained from the following Green formula for the equation (27) that corresponds to equations $l_k[y] = m_k[f]$

$$\begin{split} \int_{\alpha}^{\beta} \left(W\left(t, \mathbf{l}_{1}^{*}, \mathbf{m}_{1}\right) F_{1}\left(t, \mathbf{l}_{1}^{*}, \mathbf{m}_{1}\right), \vec{y}_{2}\left(t, \mathbf{l}_{2}, \mathbf{m}_{2}, f_{2}\right) \right) dt - \\ & - \int_{\alpha}^{\beta} \left(W\left(t, \mathbf{l}_{2}^{*}, \mathbf{m}_{2}^{*}\right) \vec{y}_{1}\left(t, \mathbf{l}_{1}, \mathbf{m}_{1}, f_{1}\right), F_{2}\left(t, \mathbf{l}_{2}^{*}, \mathbf{m}_{2}\right) \right) dt + \\ & + \int_{\alpha}^{\beta} \left(\left(H\left(t, \mathbf{l}_{1}\right) - H\left(t, \mathbf{l}_{2}^{*}\right) \right) \vec{y}_{1}\left(t, \mathbf{l}_{1}, \mathbf{m}_{1}, f_{1}\right), \vec{y}_{2}\left(t, \mathbf{l}_{2}, \mathbf{m}_{2}f_{2}\right) \right) dt - \\ & - \int_{\alpha}^{\beta} \frac{i}{2} \left\{ \left(\left(S\left(t, \mathbf{l}_{1}\right) - Q^{*}\left(t, \mathbf{l}_{2}\right) \right) \vec{y}_{1}'\left(t, \mathbf{l}_{1}, \mathbf{m}_{1}, f_{1}\right), \vec{y}_{2}\left(t, \mathbf{l}_{2}, \mathbf{m}_{2}, f_{2}\right) \right) - \\ & - \left(\left(Q\left(t, \mathbf{l}_{1}\right) - S^{*}\left(t, \mathbf{l}_{2}\right) \right) \vec{y}_{1}\left(t, \mathbf{l}_{1}, \mathbf{m}_{1}, f_{1}\right), \vec{y}_{2}'\left(t, \mathbf{l}_{2}, \mathbf{m}_{2}, f_{2}\right) \right) \right\} dt = \\ & = \left(\frac{i}{2} \left(Q\left(t, \mathbf{l}_{1}\right) + Q^{*}\left(t, \mathbf{l}_{2}\right) \right) \vec{y}_{1}\left(t, \mathbf{l}_{1}, \mathbf{m}_{1}, f_{1}\right), \vec{y}_{2}\left(t, \mathbf{l}_{2}, \mathbf{m}_{2}, f_{2}\right) \right) \right|_{\alpha}^{\beta}. \quad (44)$$

Let $r = s_k = 2n$. For convenience by $F_k(t, \mathbf{l}_k^*)$, $Y_k(t, \mathbf{l}_k)$ we denote $F_k(t, \mathbf{l}_k^*, \mathbf{m}_k)$, $Y_k(t, \mathbf{l}_k, \mathbf{m}_k)$ correspondingly. Then in view of (8), (28), (29), (34), (43) one has:

$$(W(t, l_1^*, m_1) F_1(t, l_1^*), \vec{y}_2(t, l_2, m_2, f_2)) = m_1 \{f_1, y_2\} + \left(H(t, l_1) col \{0, ..., 0, f_1^{[n]}(t | m_1)\}, col \{0, ..., 0, y_2^{[n]}(t | l_2) - y_2^{[n]}(t | l_1^*) - f_2^{[n]}(t | m_2)\} \right);$$

$$(45)$$

$$(W(t, l_{2}^{*}, m_{2}^{*}) \vec{y}_{1}(t, l_{1}, m_{1}, f_{1}), F_{2}(t, l_{2}^{*})) = m_{2}^{*} \{y_{1}, f_{2}\} + (H(t, l_{2}^{*}) col \{0, ..., 0, y_{1}^{[n]}(t | l_{1}) - y_{1}^{[n]}(t | l_{2}^{*}) - f_{1}^{[n]}(t | m_{1})\}, col \{0, ..., 0, f_{2}^{[n]}(t | m_{2})\});$$

$$(46)$$

$$\left(\left(H\left(t, l_{1}\right) - H\left(t, l_{2}^{*}\right) \right) \vec{y}_{1}\left(t, l_{1}, \mathbf{m}_{1}, f_{1}\right), \vec{y}_{2}\left(t, l_{2}, \mathbf{m}_{2}, f_{2}\right) \right) = -\left(l_{1} - l_{2}^{*}\right) \left\{ y_{1}, y_{2} \right\} - \left(\left(H\left(t, l_{1}\right) - H\left(t, l_{2}^{*}\right) \right) Y_{1}\left(t, l_{1}\right), \mathcal{F}_{2}\left(t, \mathbf{m}_{2}\right) \right) - \left(\left(H\left(t, l_{1}\right) - H\left(t, l_{2}^{*}\right) \right) \mathcal{F}_{1}\left(t, \mathbf{m}_{1}\right), Y_{2}\left(t, l_{2}\right) \right) + \left(\left(H\left(t, l_{1}\right) - H\left(t, l_{2}^{*}\right) \right) \operatorname{col}\left\{ 0, \ldots, 0, f_{1}^{\left[n\right]}\left(t \mid \mathbf{m}_{1}\right) \right\}, \operatorname{col}\left\{ 0, \ldots, 0, f_{2}^{\left[n\right]}\left(t \mid \mathbf{m}_{2}\right) \right\} \right).$$
(47)

where $\mathcal{F}_k(t, \mathbf{m}_k)$ are the analogs of (37). Let us denote by p_j^k, q_j^k, s_j^k the coefficients of l_j . Then in view of (8)

$$\left(H\left(t, \mathbf{l}_{1}\right) col\left\{0, \dots, 0, f_{1}^{[n]}\left(t \mid \mathbf{m}_{1}\right)\right\}, col\left\{0, \dots, 0, y_{2}^{[n]}\left(t \mid \mathbf{l}_{2}\right) - y_{2}^{[n]}\left(t \mid \mathbf{l}_{1}^{*}\right)\right\} \right) = \\ = \left(\left(p_{n}^{1}\right)^{-1} f_{1}^{[n]}\left(t \mid \mathbf{m}_{1}\right), y_{2}^{[n]}\left(t \mid \mathbf{l}_{2}\right) - y_{2}^{[n]}\left(t \mid \mathbf{l}_{1}^{*}\right) \right), \quad (48)$$

and

$$\left(col \left\{ 0, \dots, 0, y_1^{[n]}(t \mid l_1) - y_1^{[n]}(t \mid l_2^*) \right\}, H(t, l_2) col \left\{ 0, \dots, 0, f_2^{[n]}(t \mid m_2) \right\} \right) = \\ = \left(y_1^{[n]}(t \mid l_1) - y_1^{[n]}(t \mid l_2^*), (p_n^2)^{-1} f_2^{[n]}(t \mid m_2) \right).$$
(49)

On the another hand in view of (8), (12) we have

$$-\left(\left(H\left(t,\mathbf{l}_{1}\right)-H\left(t,\mathbf{l}_{2}^{*}\right)\right)Y_{1}\left(t,\mathbf{l}_{1}\right),\mathcal{F}_{2}\left(t,\mathbf{m}_{2}\right)\right) = \\ = -\left(\left(i(p_{n}^{1})^{-1}q_{n}^{1}/2-i(p_{n}^{2*})^{-1}s_{n}^{2*}/2\right)y_{1}^{(n-1)}+\left((p_{n}^{1})^{-1}-(p_{n}^{2*})^{-1}\right)y_{1}^{[n]}\left(t\left|\mathbf{l}_{1}\right),f_{2}^{[n]}\left(t\left|\mathbf{m}_{2}\right)\right)\right) = \\ = \left(\left(p_{n}^{2*}\right)^{-1}\left(y_{1}^{[n]}\left(t\left|\mathbf{l}_{1}\right.\right)-y_{1}^{[n]}\left(t\left|\mathbf{l}_{2}^{*}\right.\right)\right),f_{2}^{[n]}\left(t\left|\mathbf{m}_{2}\right.\right)\right),$$
(50)

where the last equality is a corollary of (12) and its following modification:

$$(p_n^1)^{-1}y_1^{[n]}(t|\mathbf{l}_1) = y_1^{(n)} - \frac{i}{2}(p_n^1)^{-1}q_n^1y_1^{(n-1)}$$

Analogously it can be proved that

$$\left(\left(H\left(t, \mathbf{l}_{1}\right) - H\left(t, \mathbf{l}_{2}^{*}\right) \right) \mathcal{F}_{1}\left(t, \mathbf{m}_{1}\right), Y_{2}\left(t, \mathbf{l}_{2}\right) \right) = \\ = \left(f_{1}^{[n]}\left(t \mid \mathbf{m}_{1}\right), \left(p_{n}^{1*}\right)^{-1}\left(y_{2}^{[n]}\left(t \mid \mathbf{l}_{2}\right) - y_{2}^{[n]}\left(t \mid \mathbf{l}_{1}^{*}\right)\right) \right).$$
 (51)

Comparing (44)–(51) we get (42) since the last \int_{α}^{β} in the left-hand-side of (44) is equal to zero if r = 2n in view of (7).

For s < r = 2n the proof of (42) easy follows from (26), (28), (34), (43), (44) in view of

footnote 1. Now let r = 2n+1. Then the last \int_{α}^{β} in the left-hand-side of (44) is equal to $\int_{\alpha}^{\beta} (l_{2n+1}^1 - l_{2n+1}^{2*}) \{y_1, y_2\} dt$. Hence the proof of (42) for $s \le 2n < r = 2n+1$ follows from (17), (26), (28), (34), (43), (44). Theorem 1.3 is proved.

Remark 1.1. In view of Lemmas 1.2, 1.3 all results of this item are valid if the condition of parity of s is changed by the condition $s \leq 2\left\lceil \frac{r}{2} \right\rceil$.

2. Characteristic operator

We consider an operator differential equation in separable Hilbert space \mathcal{H}_1 :

$$\frac{i}{2}\left(\left(Q\left(t\right)x\left(t\right)\right)' + Q^{*}\left(t\right)x'\left(t\right)\right) - H_{\lambda}\left(t\right)x\left(t\right) = W_{\lambda}\left(t\right)F\left(t\right), \quad t \in \bar{\mathcal{I}},$$
(52)

where Q(t), $[\Re Q(t)]^{-1}$, $H_{\lambda}(t) \in B(\mathcal{H}_1)$, $Q(t) \in C^1(\bar{\mathcal{I}}, B(\mathcal{H}_1))$; the operator function $H_{\lambda}(t)$ is continuous in t and is Nevanlinna's in λ . Namely the following condition holds:

(A) The set $\mathcal{A} \supseteq \mathbb{C} \setminus \mathbb{R}^1$ exists, any its point have a neighbourhood independent of $t \in \mathcal{I}$, in this neighbourhood $H_{\lambda}(t)$ is analytic $\forall t \in \overline{\mathcal{I}}$; $\forall \lambda \in \mathcal{A} H_{\lambda}(t) = H^*_{\overline{\lambda}}(t) \in C(\overline{\mathcal{I}}, B(\mathcal{H}_1))$; the weight $W_{\lambda}(t) = \Im H_{\lambda}(t) / \Im \lambda \ge 0 (\Im \lambda \neq 0)$.

In view of [22] $\forall \mu \in \mathcal{A} \cap \mathbb{R}^1$: $W_{\mu}(t) = \partial H_{\lambda}(t) / \partial \lambda|_{\lambda=\mu}$ is Bochner locally integrable in the uniform operator topology.

For convenience we suppose that $0 \in \overline{\mathcal{I}}$ and we denote $\Re Q(0) = G$.

Let $X_{\lambda}(t)$ be the operator solution of homogeneous equation (52) satisfying the initial condition $X_{\lambda}(0) = I$, where I is an identity operator in \mathcal{H}_1 . Since $H_{\lambda}(t) = H^*_{\overline{\lambda}}(t)$ then

$$X_{\overline{\lambda}}^*(t)[\Re Q(t)]X_{\lambda}(t) = G, \ \lambda \in \mathcal{A}.$$
(53)

For any $\alpha, \beta \in \overline{\mathcal{I}}, \alpha \leq \beta$ we denote $\Delta_{\lambda}(\alpha, \beta) = \int_{\alpha}^{\beta} X_{\lambda}^{*}(t) W_{\lambda}(t) X_{\lambda}(t) dt$, $N = \{h \in \mathcal{H}_{1} | h \in Ker\Delta_{\lambda}(\alpha, \beta) \forall \alpha, \beta\}, P$ is the ortho-projection onto N^{\perp} . N is independent

 $N = \{h \in \mathcal{H}_1 \mid h \in Ker\Delta_\lambda(\alpha, \beta) \; \forall \alpha, \beta\}, P \text{ is the ortho-projection onto } N^{\perp}. N \text{ is independent}$ of $\lambda \in \mathcal{A}$ [22].

For $x(t) \in \mathcal{H}_1$ or $x(t) \in B(\mathcal{H}_1)$ we denote $U[x(t)] = ([\Re Q(t)] x(t), x(t))$ or $U[x(t)] = x^*(t) [\Re Q(t)] x(t)$ respectively.

As in [21, 22] we introduce the following

Definition 2.1. An analytic operator-function $M(\lambda) = M^*(\overline{\lambda}) \in B(\mathcal{H}_1)$ of non-real λ is called a characteristic operator of equation (52) on \mathcal{I} , if for $\Im \lambda \neq 0$ and for any \mathcal{H}_1 - valued vector-function $F(t) \in L^2_{W_{\lambda}}(\mathcal{I})$ with compact support the corresponding solution $x_{\lambda}(t)$ of equation (52) of the form

$$x_{\lambda}(t,F) = \mathcal{R}_{\lambda}F = \int_{\mathcal{I}} X_{\lambda}(t) \left\{ M(\lambda) - \frac{1}{2}sgn(s-t)(iG)^{-1} \right\} X_{\overline{\lambda}}^{*}(s) W_{\lambda}(s) F(s) ds$$
(54)

satisfies the condition

$$(\Im\lambda)\lim_{(\alpha,\beta)\uparrow\mathcal{I}} \left(U\left[x_{\lambda}\left(\beta,F\right) \right] - U\left[x_{\lambda}\left(\alpha,F\right) \right] \right) \le 0 \quad (\Im\lambda\neq0) \,. \tag{55}$$

Let us note that in [22] characteristic operator was defined if $Q(t) = Q^*(t)$. Our case is equivalent to this one since equation (52) coincides with equation of (52) type with $\Re Q(t)$ instead of Q(t) and with $H_{\lambda}(t) - \frac{1}{2}\Im Q'(t)$ instead of $H_{\lambda}(t)$.

The properties of characteristic operator and sufficient conditions of the characteristic operators existence are obtained in [21, 22].

In the case $\dim \mathcal{H}_1 < \infty$, $Q(t) = \mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$, $-\infty < a = c$ the description of characteristic operators was obtained in [31] (the results of [31] were specified and supplemented in [23]). In the case $\dim \mathcal{H}_1 = \infty$ and \mathcal{I} is finite the description of characteristic operators was obtained in [22]. These descriptions are obtained under the condition that

$$\exists \lambda_0 \in \mathcal{A}, \ [\alpha, \beta] \subseteq \overline{\mathcal{I}} : \ \Delta_{\lambda_0}(\alpha, \beta) \gg 0.$$
(56)

Definition 2.2. [21, 22] Let $M(\lambda)$ be the characteristic operator of equation (52) on \mathcal{I} . We say that the corresponding condition (55) is separated for nonreal $\lambda = \mu_0$ if for any \mathcal{H}_1 -valued vector function $f(t) \in L^2_{W_{\mu_0}(t)}(\mathcal{I})$ with compact support the following inequalities holds simultaneously for the solution $x_{\mu_0}(t)$ (54) of equation (52):

$$\lim_{\alpha \downarrow a} \Im \mu_0 U\left[x_{\mu_0}\left(\alpha\right)\right] \ge 0, \quad \lim_{\beta \uparrow b} \Im \mu_0 U\left[x_{\mu_0}\left(\beta\right)\right] \le 0.$$
(57)

Theorem 2.1. [21, 22] (see also [34]) Let P = I, $M(\lambda)$ be the characteristic operator of equation (52), $\mathcal{P}(\lambda) = iM(\lambda)G + \frac{1}{2}I$, so that we have the following representation

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}.$$
(58)

Then the condition (55) corresponding to $M(\lambda)$ is separated for $\lambda = \mu_0$ if and only if the operator $\mathcal{P}(\mu_0)$ is the projection, i.e.

$$\mathcal{P}\left(\mu_{0}\right) = \mathcal{P}^{2}\left(\mu_{0}\right). \tag{59}$$

Definition 2.3. [21, 22] If the operator-function $M(\lambda)$ of the form (58) is the characteristic operator of equation (52) on \mathcal{I} and, moreover, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $\mathcal{P}(\lambda)$ is called a characteristic projection of equation (52) on \mathcal{I} .

The properties of characteristic projections and sufficient conditions for their existence are obtained in [22]. Also [22] contains the description of characteristic projections and abstract an analogue of Theorem 2.1.

The following statement gives necessary and sufficient conditions for existence of characteristic operator, which corresponds to such separated boundary conditions that corresponding boundary condition in regular point is self-adjoint. This statement follows from Theorem 2.1.

Let us denote \mathcal{H}_+ (\mathcal{H}_-) the invariant subspace of operator G, which corresponds to positive (negative) part of $\sigma(G)$.

Theorem 2.2. Let $-\infty < a$. If P = I then for existence of characteristic operator $M(\lambda)$ of equation (52) on (a, b) such that

$$\exists \mu_0 \in \mathbb{C} \setminus \mathbb{R}^1 : \ U[x_{\mu_0}(a,F)] = U[x_{\bar{\mu_0}}(a,F)] = 0$$

$$\tag{60}$$

(and therefore condition (55) is separated on $\lambda = \mu_0$, $\lambda = \bar{\mu}_0$) it is necessary that

$$\dim \mathcal{H}_{+} = \dim \mathcal{H}_{-} \tag{61}$$

(in (60) $x_{\lambda}(t, F)$ is a solution (54) of (52) which corresponds to characteristic operator $M(\lambda)$, $L^2_{w_{\mu_0}(t)}(a, b) \ni F = F(t)$ is any \mathcal{H}_1 -valued vector-function with compact support). If condition (56) holds then condition (61) is also sufficient for the existence of such characteristic operator.

Proof. Necessity. Since P = I we obtain

$$U[X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))] = U[X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))] = 0$$
(62)

in view of the proof of $n^{\circ}2^{\circ}$ of Theorem 1.1 from [22].

Let for definiteness $\Im \mu_0 > 0$. Then in view of Theorem 2.4 and formula (1.69) from [22], (59), (62) and the fact that

$$\Im\lambda(X_{\lambda}^{*}(a)\Re Q(a)X_{\lambda}(a) - G) \le 0, \ \lambda \in \mathcal{A}$$
(63)

we conclude that $X_{\mu_0}(a)(I-\mathcal{P}(\mu_0))\mathcal{H}_1$ and $X_{\bar{\mu}_0}(a)(I-\mathcal{P}(\bar{\mu}_0))\mathcal{H}_1$ are correspondingly maximal $\Re Q(a)$ -nonnegative and maximal $\Re Q(a)$ -nonpositive subspaces which are $\Re Q(a)$ -neutral and which are $\Re Q(a)$ -orthogonal in view of Remark 3.2 from [22], Theorem 2.1 and (53). Hence

$$(X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))\mathcal{H}_1)^{[\perp]} = X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))\mathcal{H}_1$$

in view of [3, p.73] (here by $[\bot]$ we denote $\Re Q(a)$ -orthogonal complement). Therefore $X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))\mathcal{H}_1$ is hypermaximal $\Re Q(a)$ -neutral subspace in view of [3, p.43]. Thus we obtain that in view of [3, p.42] that dim $\mathcal{H}_+(a) = \dim \mathcal{H}_-(a)$, where $\mathcal{H}_{\pm}(a)$ are analogs of \mathcal{H}_{\pm} for $\Re Q(a)$. In view of (63) $X_{\mu_0}^{-1}(a)\mathcal{H}_+(a)$ and $X_{\bar{\mu}_0}^{-1}(a)\mathcal{H}_-(a)$ are correspondingly maximal uniformly *G*-positive and maximal uniformly *G*-negative subspaces. Therefore \mathcal{H}_1 is equal to the direct and *G*-orthogonal sum of these subspaces in view of (53) and [3, p.75]. Hence we obtain (61) in view of the law of inertia [3, p.54].

Sufficiency follows from Theorem 4.4. from [22]. Theorem is proved.

It is obvious that in Theorem 2.2 the point a can be replaced by the point b if $b < \infty$, but cannot be replaced by the point b if $b = \infty$ as the example of operator id/dt on the semi-axis shows. Also this example shows that condition (60) is not necessary for the fulfilment of the condition $U[x_{\mu_0}(a, F)] = 0$ only.

In the case of self-adjoint boundary conditions the analogue of Theorem 2.2 for regular differential operators in space of vector-functions was proved in [32] (see also [34]). For finite canonical systems depending on spectral parameter in a linear manner such analogue was proved in [29]. These analogs were obtained in a different way comparing with Theorem 2.2.

Let $\mathcal{H}_1 = \mathcal{H}^{2n}$, Q(t) = J/i (7), a = c and condition (56) hold. Let condition (55) be separated and $\mathcal{P}(\lambda)$ be a corresponding characteristic projection. In view of [22, p. 469] the Nevanlinna pair $\{-a(\lambda), b(\lambda)\}, a, b \in B(\mathcal{H}^n)$ (see for example [13]) and Weyl function $m(\lambda) \in B(\mathcal{H}^n)$ of equation (52) on (c, b) [22] exist such that

$$\mathcal{P}\left(\lambda\right) = \begin{pmatrix} I_n \\ m\left(\lambda\right) \end{pmatrix} \left(b^*\left(\bar{\lambda}\right) - a^*\left(\bar{\lambda}\right)m\left(\lambda\right)\right)^{-1} \left(a_2^*\left(\bar{\lambda}\right), -a_1^*\left(\bar{\lambda}\right)\right), \tag{64}$$

$$I - \mathcal{P}(\lambda) = \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} (b(\lambda) - m(\lambda)a(\lambda))^{-1} (-m(\lambda), I_n),$$
(65)

$$(b^*(\overline{\lambda}) - a^*(\overline{\lambda}) m(\lambda))^{-1}, (b(\lambda) - m(\lambda) a(\lambda))^{-1} \in B(\mathcal{H}^n)$$

Conversely $\mathcal{P}(\lambda)$ (64) is a characteristic projection for any Nevanlinna pair $(-a(\lambda), b(\lambda))$ and any Weyl function $m(\lambda)$ of equation (52) on (c, b).

Let domain $D \subseteq \mathbb{C}_+$ be such that $\forall \lambda \in D : 0 \in \rho(a(\lambda) - ib(\lambda))$ (for example $D = \mathbb{C}_+$ if $\exists \lambda_{\pm} \in \mathbb{C}_{\pm}$ such that $a^*(\lambda_{\pm}) b(\lambda_{\pm}) = b^*(\lambda_{\pm}) a(\lambda_{\pm})$). Let domain D_1 be symmetric to D with respect to real axis. Then in view of (64), (65) and Corrolary 3.1 from [22] operator $\mathcal{R}_{\lambda}F$ (54) for $\lambda \in D \bigcup D_1$ can be represented in the following form with using the operator solution $U_{\lambda}(t) \in B(\mathcal{H}^n, \mathcal{H}^{2n})$ of equation (52), (F = 0) satisfying accumulative (or dissipative) initial condition and operator solution $V_{\lambda}(t) \in B(\mathcal{H}^n, \mathcal{H}^{2n})$ of Weyl type of the same equation.

Remark 2.1. Let $\lambda \in D \bigcup D_1$. Then

$$\mathcal{R}_{\lambda}F = \int_{a}^{t} V_{\lambda}(t) U_{\overline{\lambda}}^{*}(s) W_{\lambda}(s) F(s) ds + \int_{t}^{b} U_{\lambda}(t) V_{\overline{\lambda}}^{*}(s) W_{\lambda}(s) F(s) ds$$

Here

$$U_{\lambda}(t) = X_{\lambda}(t) \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}, \quad V_{\lambda}(t) = X_{\lambda}(t) \begin{pmatrix} b(\lambda) \\ -a(\lambda) \end{pmatrix} K^{-1}(\lambda) + U_{\lambda}(t) m_{a,b}(\lambda), \quad (66)$$

where

$$K(\lambda) = a^*(\bar{\lambda}) a(\lambda) + b^*(\bar{\lambda}) b(\lambda), \ K^{-1}(\lambda) \in B(\mathcal{H}^n),$$
(67)

$$m_{a,b}\left(\lambda\right) = m_{a,b}^{*}\left(\bar{\lambda}\right) = K^{-1}\left(\lambda\right) \left(a^{*}\left(\bar{\lambda}\right) + b^{*}\left(\bar{\lambda}\right)m\left(\lambda\right)\right) \left(b^{*}\left(\bar{\lambda}\right) - a^{*}\left(\bar{\lambda}\right)m\left(\lambda\right)\right)^{-1}, \quad (68)$$

$$V_{\lambda}(t) h \in L^{2}_{W_{\lambda}(t)}(c,b) \,\forall h \in \mathcal{H}^{n}.$$
(69)

Moreover if $\exists \lambda_0 \in \mathbb{C} \setminus \mathbb{R}^1$ such that $a(\lambda_0) = a(\overline{\lambda}_0)$, $b(\lambda_0) = b(\overline{\lambda}_0)$ then we can set $D = \mathbb{C}_+$ and

$$\int_{\mathcal{I}} V_{\lambda}^{*}(t) W_{\lambda}(t) V(t) dt \leq \frac{\Im m_{a,b}(\lambda)}{\Im \lambda} (\Im \lambda \neq 0).$$

For the construction of solutions of Weyl type and descriptions of Weyl function in various situation see [1, 22] and references in [1].

Let us consider operator differential expression l_{λ} of (6) type with coefficients $p_j = p_j(t, \lambda)$, $q_j = q_j(t, \lambda)$, $s_j = s_j(t, \lambda)$ and of order r. Let $-l_{\lambda}$ depends on λ in Nevanlinna manner. Namely, from now on the following condition holds:

(B) The set $\mathcal{B} \supseteq \mathbb{C} \setminus \mathbb{R}^1$ exists, any its points have a neighbourhood independent on $t \in \overline{\mathcal{I}}$, in this neighbourhood coefficients $p_j = p_j(t,\lambda)$, $q_j = q_j(t,\lambda)$, $s = s_j(t,\lambda)$ of the expression l_{λ} are analytic $\forall t \in \overline{\mathcal{I}}$; $\forall \lambda \in \mathcal{B}$, $p_j(t,\lambda)$, $q_j(t,\lambda)$, $s_j(t,\lambda) \in C^j(\overline{\mathcal{I}}, \mathcal{B}(\mathcal{H}))$ and

$$p_n^{-1}(t,\lambda) \in B(\mathcal{H})(r=2n), (q_{n+1}(t,\lambda)+s_{n+1}(t,\lambda))^{-1} \in B(\mathcal{H})(r=2n+1), \ t \in \bar{\mathcal{I}};$$
(70)

these coefficients satisfy the following conditions

$$p_{j}(t,\lambda) = p_{j}^{*}(t,\bar{\lambda}), \ q_{j}(t,\lambda) = s_{j}^{*}(t,\bar{\lambda}), \ \lambda \in \mathcal{B}$$

$$(71)$$

$$((71) \iff l_{\lambda} = l_{\bar{\lambda}}^* \iff H(t, l_{\lambda}) = H(t, l_{\bar{\lambda}}^*), \lambda \in \mathcal{B});$$

$$\forall h_0, \dots, h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H} :$$

$$\frac{\Im\left(\sum_{j=0}^{\left[r/2\right]} \left(p_j\left(t,\lambda\right)h_j, h_j\right) + \frac{i}{2} \sum_{j=1}^{\left[\frac{r+1}{2}\right]} \left\{ \left(s_j\left(t,\lambda\right)h_j, h_{j-1}\right) - \left(q_j\left(t,\lambda\right)h_{j-1}, h_j\right) \right\} \right)}{\Im\lambda} \leq 0,$$

$$t \in \bar{\mathcal{I}}, \ \Im\lambda \neq 0.$$
(72)

Therefore the order of expression $\Im l_{\lambda}$ is even and therefore if r = 2n + 1 is odd, then q_{m+1}, s_{m+1} are independent on λ and $s_{n+1} = q_{n+1}^*$.

Condition (72) is equivalent to the condition: $(\Im l_{\lambda}) \{f, f\} / \Im \lambda \leq 0, t \in \overline{I}, \Im \lambda \neq 0.$ Hence $W\left(t, l_{\lambda}, -\frac{\Im l_{\lambda}}{\Im \lambda}\right) = \frac{\Im H(t, l_{\lambda})}{\Im \lambda} \geq 0, t \in \overline{I}, \Im \lambda \neq 0$ due to Lemma 1.1 and Theorem 1.2 and therefore $H\left(t, l_{\lambda}\right)$ satisfy condition (A) with $\mathcal{A} = \mathcal{B}$. Therefore $\forall \mu \in \mathcal{B} \cap \mathbb{R}^1$ $W(t, l_{\mu}, -\frac{\Im l_{\mu}}{\Im \mu}) = \frac{\partial H(t, l_{\lambda})}{\partial \lambda}\Big|_{\lambda=\mu}$ is Bochner locally integrable in uniform operator topology. Here in view of (8), (16) $\forall \mu \in \mathcal{B} \cap \mathbb{R}^1 \exists \frac{\Im l_{\mu}}{\Im \mu} \stackrel{def}{=} \frac{\Im l_{\mu+i0}}{\Im(\mu+i0)} = \frac{\partial l_{\lambda}}{\partial \lambda}\Big|_{\lambda=\mu}$, where the coefficients $\frac{\partial p_j(t,\mu)}{\partial \lambda}$, $\frac{\partial q_j(t,\mu)}{\partial \lambda}, \frac{\partial s_j(t,\mu)}{\partial \lambda}$ of expression $\partial l_{\mu}/\partial \mu$ are Bochner locally integrable in the uniform operator topology.

Let us consider in $\mathcal{H}_1 = \mathcal{H}^r$ the equation

$$\frac{i}{2}\left(\left(Q\left(t,l_{\lambda}\right)\vec{y}\left(t\right)\right)'+Q^{*}\left(t,l_{\lambda}\right)\vec{y}'\left(t\right)\right)-H\left(t,l_{\lambda}\right)\vec{y}\left(t\right)=W\left(t,l_{\lambda}-\frac{\Im l_{\lambda}}{\Im\lambda}\right)F\left(t\right).$$
(73)

This equation is an equation of (52) type due to (18) and Lemma 1.1. Equation (5) is equivalent to equation (73) with $F(t) = F\left(t, l_{\bar{\lambda}}, -\frac{\Im l_{\lambda}}{\Im \lambda}\right)$ due to Theorem 1.1.

Definition 2.4. Every characteristic operator of equation (73) corresponding to the equation (5) is said to be a characteristic operator of equation (5) on \mathcal{I} .

Let *m* be the same as in $n^{\circ}1$ differential expression of even order $s \leq r$ with operator coefficients $\tilde{p}_j(t) = \tilde{p}_j^*(t)$, $\tilde{q}_j(t)$, $\tilde{s}_j(t) = \tilde{q}_j^*(t)$ that are independent on λ . Let

$$\forall h_0, \dots, h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H}: \ 0 \le \sum_{j=0}^{s/2} \left(\tilde{p}_j\left(t\right) h_j, h_j \right) + \Im \sum_{j=1}^{s/2} \left(\tilde{q}_j\left(t\right) h_{j-1}, h_j \right) \le \\ \le -\frac{\Im \left(\sum_{j=0}^{\left[r/2\right]} \left(p_j\left(t,\lambda\right) h_j, h_j \right) + \frac{i}{2} \sum_{j=1}^{\left[\frac{r+1}{2}\right]} \left(\left(s_j\left(t,\lambda\right) h_j, h_{j-1} \right) - \left(q_j\left(t,\lambda\right) h_{j-1}, h_j \right) \right) \right) \\ \le -\frac{\Im \lambda}{t \in \bar{\mathcal{I}}, \ \Im \lambda \neq 0.$$
 (74)

Condition (74) is equivalent to the condition: $0 \leq m\{f, f\} \leq -(\Im l_{\lambda})\{f, f\}/\Im \lambda, t \in \overline{\mathcal{I}}, \Im \lambda \neq 0$. Hence

$$0 \le W(t, l_{\lambda}, m) \le W\left(t, l_{\lambda}, -\frac{\Im l_{\lambda}}{\Im \lambda}\right) = \frac{\Im H(t, l_{\lambda})}{\Im \lambda} \quad t \in \bar{\mathcal{I}}, \ \Im \lambda \ne 0$$
(75)

due to Theorem 1.2 and Lemma 1.1.

In view of Theorem 1.1 equation (1) is equivalent to the equation

$$\frac{i}{2} \left(\left(Q\left(t, l_{\lambda}\right) \vec{y}\left(t\right) \right)' + Q^{*}\left(t, l_{\lambda}\right) \vec{y}'\left(t\right) \right) - H\left(t, l_{\lambda}\right) \vec{y}\left(t\right) = W\left(t, l_{\bar{\lambda}}, m\right) F\left(t, l_{\bar{\lambda}}, m\right),$$
(76)

where $Q(t, l_{\lambda})$, $H(t, l_{\lambda})$ are defined by (7), (8), (15), (16) with l_{λ} instead of l and $W(t, l_{\bar{\lambda}}, m)$ $F(t, l_{\bar{\lambda}}, m)$ are defined by (9), (17) (26) with $l_{\bar{\lambda}}$ instead of l and $\vec{y}(t) = \vec{y}(t, l_{\lambda}, m, f)$ is defined by (28) with l_{λ} instead of l.

In some cases we will suppose additionally that $\exists \lambda_0 \in \mathcal{B}; \ \alpha, \beta \in \overline{\mathcal{I}}, \ 0 \in [\alpha, \beta], \text{ the number } \delta > 0:$

$$-\int_{\alpha}^{\beta} \left(\frac{\Im l_{\lambda_0}}{\Im \lambda_0}\right) \left\{ y\left(t,\lambda_0\right), y\left(t,\lambda_0\right) \right\} dt \ge \delta \left\| P\vec{y}\left(0,l_{\lambda_0},m,0\right) \right\|^2 \tag{77}$$

for any solution $y(t, \lambda_0)$ of (5) as $\lambda = \lambda_0$, f = 0. In view of Theorem 1.2 this condition is equivalent to the fact that for the equation (73) with F(t) = 0

 $\exists \lambda_0 \in \mathcal{A} = \mathcal{B}; \ \alpha, \beta \in \overline{\mathcal{I}}, \ 0 \in [\alpha, \beta], \text{ the number } \delta > 0:$

$$(\Delta_{\lambda_0}(\alpha,\beta)\,g,g) \ge \delta \,\|Pg\|^2\,, \quad g \in \mathcal{H}^r.$$
(78)

Therefore in view of [24] the fulfillment of (77) imply its fulfillment with $\delta(\lambda) > 0$ instead of δ for all $\lambda \in \mathcal{B}$.

Lemma 2.1. Let $M(\lambda)$ be a characteristic operator of equation (5), for which condition (77) holds with $P = I_r$, if \mathcal{I} is infinite. Let $\Im \lambda \neq 0$, \mathcal{H}^r -valued $F(t) \in L^2_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I})$ (in particular one can set $F(t) = F(t, l_{\bar{\lambda}}, m)$, where $f(t) \in C^s(\mathcal{I}, \mathcal{H})$, $m[f, f] < \infty$). Then the solution

$$x_{\lambda}(t,F) = \mathcal{R}_{\lambda}F = \int_{\mathcal{I}} X_{\lambda}(t) \left\{ M(\lambda) - \frac{1}{2} sgn(s-t)(iG)^{-1} \right\} X_{\bar{\lambda}}^{*}(s) W(s,l_{\bar{\lambda}},m) F(s) ds$$
(79)

of equation (76) with F(t) instead $F(t, l_{\bar{\lambda}}, m)$, satisfies the following inequality

$$\left\|\mathcal{\mathcal{R}}_{\lambda}F\right\|_{L^{2}_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\mathcal{I})}^{2} \leq \Im\left(\mathcal{\mathcal{R}}_{\lambda}F,F\right)_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})}/\Im\lambda, \ \Im\lambda\neq0,\tag{80}$$

where $X_{\lambda}(t)$ is the operator solution of homogeneous equation (76) such that $X_{\lambda}(0) = I_r$, $G = \mathcal{R}Q(0, l_{\lambda})$; integral (79) converges strongly if \mathcal{I} is infinite.

Proof. Let us denote

$$K(t,s,\lambda) = X_{\lambda}(t) \left\{ M(\lambda) - \frac{1}{2} sgn(s-t) (iG)^{-1} \right\} X_{\overline{\lambda}}^{*}(s)$$

If (77) holds with $P = I_r$ if \mathcal{I} is infinite, then in view of (75) and [22, p.166] there exists a locally bounded on s and on λ constant $k(s, \lambda)$ such that

$$\forall h \in \mathcal{H}^{r}: \quad \|K(t, s, \lambda) h\|_{L^{2}_{W\left(t, l_{\bar{\lambda}}, m\right)}(\mathcal{I})} \leq k(s, \lambda) \|h\|$$

$$(81)$$

Hence integral (79) converges strongly if \mathcal{I} is in finite. Let F(t) have compact support and $suppF(t) \subseteq [\alpha, \beta]$. Then in view of (42)

$$\int_{\alpha}^{\beta} \left(W\left(t, l_{\lambda}, -\frac{\Im l_{\lambda}}{\Im \lambda} \right) \mathcal{R}_{\lambda} F, \mathcal{R}_{\lambda} F \right) dt - \frac{\Im \int_{\alpha}^{\beta} \left(W\left(t, l_{\bar{\lambda}}, m\right) \mathcal{R}_{\lambda} F, F \right) dt}{\Im \lambda} = \frac{1}{2} \frac{\left(\Re Q\left(t, l_{\lambda}\right) \mathcal{R}_{\lambda} F, \mathcal{R}_{\lambda} F \right)}{\Im \lambda} \Big|_{\alpha}^{\beta} \leq 0 \quad (82)$$

where the last inequality is a corollary of $n^{\circ}2$. Theorem 1.1 from [22, p.162] and the following Lemma 2.2. Let \mathcal{F}_{λ} is the set of \mathcal{H}^r -valued function from $L^2_{W(t,l_{\bar{\lambda}},m)}(\alpha,\beta)$,

$$I_{\lambda}(\alpha,\beta) F = \int_{\alpha}^{\beta} X_{\bar{\lambda}}^{*}(t) W(t, l_{\bar{\lambda}}, m) F(t) dt, \quad F(t) \in \mathcal{F}_{\lambda}$$
(83)

Then

$$I_{\lambda}(\alpha,\beta) F \in \left\{ Ker \int_{\alpha}^{\beta} X_{\bar{\lambda}}^{*}(t) W(t,l_{\bar{\lambda}},m) X_{\bar{\lambda}}(t) dt \right\}^{\perp} \subseteq N^{\perp}.$$
(84)

Proof. Let $h \in Ker \int_{\alpha}^{\beta} X_{\bar{\lambda}}^{*}(t) W(t, l_{\bar{\lambda}}, m) X_{\bar{\lambda}}(t) dt \Rightarrow W(t, l_{\bar{\lambda}}, m) X_{\bar{\lambda}}(t) h = 0 \Rightarrow I_{\lambda}(\alpha, \beta) F \perp h.$ The second enclosure in (84) is a corollary of condition (75). Lemma 2.2 and inequality (82) are proved.

Thus Lemma 2.1 is proved if \mathcal{I} is finite. Let us prove it for infinite \mathcal{I} . Let finite intervals $(\alpha_n, \beta_n) \uparrow \mathcal{I}, \quad F_n = \chi_n F$, where χ_n - is a characteristic function of (α_n, β_n) . If $(\alpha, \beta) \subseteq (\alpha_n, \beta_n)$ then

$$\left\|\mathcal{R}_{\lambda}F_{n}\right\|_{L^{2}_{W\left(t,l_{\lambda},-\frac{\Im I_{\lambda}}{\Im\lambda}\right)}}(\alpha,\beta) \leq \frac{\left\|F'\right\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})}}{\left|\Im\lambda\right|}$$

in view of (82), (75). But local uniformly on $t: (\mathcal{R}_{\lambda}F_n)(t) \to (\mathcal{R}_{\lambda}F)(t)$, in view of (81). Hence

$$\|\mathcal{R}_{\lambda}F\|_{L^{2}_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\alpha,\beta)} \leq \frac{\|F\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})}}{|\Im\lambda|}.$$
(85)

for any finite (α, β) . Hence (85) holds with \mathcal{I} instead of (α, β) . In view of last fact $\mathcal{R}_{\lambda}F_n \to \mathcal{R}_{\lambda}F$ in $L^2_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\mathcal{I})$. Hence (80) is proved since it is proved for F_n . Lemma 2.1 is proved.

Let us notice that in view of [22] $PM(\lambda)P$ is a characteristic operator of equation (5), if $M(\lambda)$ is its characteristic operator Ocharacteristic operators $M(\lambda)$ and $PM(\lambda)P$ are equal in $B\left(L^{2}_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I}), L_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im \lambda}\right)}(\mathcal{I})\right)$. Let us notice what in view of (74) l_{λ} can be a represented in form (2) where

$$l = \Re l_i, n_\lambda = l_\lambda - l - \lambda m; \Im n_\lambda \{f, f\} / \Im \lambda \ge 0, t \in \overline{\mathcal{I}}, \, \Im \lambda \ne 0.$$
(86)

From now on we suppose that l_{λ} has a representation (2), (86) and therefore the order of n_{λ} is even.

3. Main results

We consider pre-Hilbert spaces $\overset{\circ}{H}$ and H of vector-functions $y(t) \in C_0^s(\bar{\mathcal{I}}, \mathcal{H})$ and $y(t) \in C_0^s(\bar{\mathcal{I}}, \mathcal{H})$ $C^{s}(\bar{\mathcal{I}},\mathcal{H}), m[y(t), y(t)] < \infty$ correspondingly with a scalar product

$$(f(t), g(t))_{m} = m[f(t), g(t)]$$

where m[f, g] is defined by (32) with expression m from condition (74) instead of L. Namely

$$m[f, g] = \int_{\mathcal{I}} m\{f, g\} dt,$$
(87)

where $m\left\{f, g\right\} = \sum_{j=0}^{s/2} (\tilde{p}_j(t) f^{(j)}(t), g^{(j)}(t)) + \frac{i}{2} \sum_{j=1}^{s/2} \left((\tilde{q}_j^*(t) f^{(j)}(t), g^{(j-1)}(t)) - (\tilde{q}_j(t) f^{(j-1)}(t), g^{(j)}(t)) \right).$ The null-elements of H are given by

Proposition 3.1. Let $f(t) \in H$. Then

$$m[f, f] = 0 \Leftrightarrow m[f] = f^{[s]}(t) = \dots = f^{[s/2]}(t) = 0, \ t \in \overline{\mathcal{I}}.$$

Proof. Let us denote by $m(t) \in B(\mathcal{H}^{n+1})$ the operator matrix corresponding to the quadratic form in left side of (74). Since $m(t) \ge 0$ one has

$$m[f, f] = 0 \Leftrightarrow m(t) \operatorname{col} \left\{ f(t), \dots, f^{(s/2)}, 0, \dots, 0 \right\} = 0 \Leftrightarrow f^{[s]}(t) = \dots = f^{[s/2]} = 0$$

Example 3.1. Let dim $\mathcal{H} = 1$, s = 2, $\tilde{p}_1(t) > 0$, $|\tilde{q}_1(t)|^2 = 4\tilde{p}_1(t)\tilde{p}_0(t)$. Then for expression *m* the first inequality (74) holds and $m\{f_0, f_0\} \equiv 0$ for $f_0(t) = \exp\left(\frac{i}{2}\int_0^t \tilde{q}_1/\tilde{p}_1 dt\right) \neq 0$ in view of Proposition 3.1.

By $L_m^{\circ}(\mathcal{I})$ and $L_m^2(\mathcal{I})$ we denote the completions of spaces $\overset{\circ}{H}$ and H in the norm $\|\bullet\|_m =$ $\sqrt{(\bullet, \bullet)_m}$ correspondingly. By $\overset{\circ}{P}$ we denote the orthoprojection in $L^2_m(\mathcal{I})$ onto $\overset{\circ}{L^2_m}(\mathcal{I})$.

Theorem 3.1. Let $M(\lambda)$ be a characteristic operator of equation (5), for which the condition (77) with $P = I_r$ holds if \mathcal{I} is infinite. Let $\Im \lambda \neq 0$, $f(t) \in H$ and

$$col \{y_j(t,\lambda,f)\} = \int_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} sgn(s-t)(iG)^{-1} \right\} X_{\bar{\lambda}}^*(s) W(s, l_{\bar{\lambda}}, m) F(s, l_{\bar{\lambda}}, m) \, ds, \, y_j \in \mathcal{H}$$
(88)

be a solution of equation (76), that corresponds to equation (1), where $X_{\lambda}(t)$ is the operator solution of homogeneous equation (76) such that $X_{\lambda}(0) = I_r$; $G = \Re Q(0, l_{\lambda})$ (if \mathcal{I} is infinite integral (88) converges strongly). Then the first component of vector function (88) is a solution of equation (1). It defines densely defined in $L^2_m(\mathcal{I})$ integro-differential operator

$$R(\lambda) f = y_1(t,\lambda,f), \quad f \in H$$
(89)

which has the following properties after closing 1°

$$R^*(\lambda) = R(\bar{\lambda}), \quad \Im \lambda \neq 0 \tag{90}$$

 2°

$$R(\lambda)$$
 is holomorphic on $\mathbb{C} \setminus \mathbb{R}^1$ (91)

3°

$$\|R(\lambda)f\|_{L^{2}_{m}(\mathcal{I})}^{2} \leq \frac{\Im(R(\lambda)f,f)_{L^{2}_{m}(\mathcal{I})}}{\Im\lambda}, \ \Im\lambda \neq 0, \ f \in L^{2}_{m}(\mathcal{I})$$

$$(92)$$

Let us notice that the definition of the operator $R(\lambda)$ is correct. Indeed if $f(t) \in H$, m[f, f] = 0, then $R(\lambda) f \equiv 0$ since $W(t, l_{\bar{\lambda}}, m) F(t, l_{\bar{\lambda}}, m) \equiv 0$ due to (34), (75).

Proof. In view of Lemma 2.1 integral (88) converges strongly if \mathcal{I} is infinite. In view of Theorem 1.1 $y_1(t, \lambda, f)$ (89) is a solution of equation (1).

In view of (74), (35)

$$\|R(\lambda)f\|_{L^{2}_{m}(\alpha,\beta)} - \frac{\Im(R(\lambda)f,f)_{L^{2}_{m}(\alpha,\beta)}}{\Im\lambda} \leq \|R(\lambda)f\|_{L^{2}_{-\frac{\Im l_{\lambda}}{\Im\lambda}}(\alpha,\beta)} - \frac{\Im(R(\lambda)f,f)_{L^{2}_{m}(\alpha,\beta)}}{\Im\lambda} = \\ = \|\mathcal{R}_{\lambda}F(t,l_{\bar{\lambda}},m)\|_{L^{2}_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\alpha,\beta)} - \frac{\Im(\mathcal{R}_{\lambda}F(t,l_{\bar{\lambda}},m),F(t,l_{\bar{\lambda}},m)_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\alpha,\beta)}}{\Im\lambda}.$$
(93)

In view of Lemma 2.1 a nonnegative limit of the right-hand-side of (93) exists, when $(\alpha, \beta) \uparrow \mathcal{I}$. Hence (92) is proved.

Let \mathcal{H}^r -valued $F(t) \in L^2_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I})$. Then in view of (75), Lemma 2.1, (19) one has

$$\|\mathcal{R}_{\lambda}F\|_{L^{2}_{W(t,l_{\lambda},m)}(\mathcal{I})}^{2} \leq \|\mathcal{R}_{\lambda}F\|_{L^{2}_{W(t,l_{\lambda},-\frac{\Im l}{\Im\lambda})}(\mathcal{I})}^{2} \leq \frac{\Im(\mathcal{R}_{\lambda}F,F)_{L^{2}_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I})}}{\Im\lambda},$$
(94)

$$\|\mathcal{R}_{\lambda}F\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})}^{2} \leq \|\mathcal{R}_{\lambda}F\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\mathcal{I})} = \|\mathcal{R}_{\lambda}F\|_{L^{2}_{W\left(t,l_{\lambda},-\frac{\Im l_{\lambda}}{\Im\lambda}\right)}(\mathcal{I})}.$$
(95)

In view of (94), (95) we have

$$\|\mathcal{R}_{\lambda}F\|_{L^{2}_{W(t,l_{\lambda},m)}(\mathcal{I})} \leq \|F\|_{L^{2}_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I})} / |\Im\lambda|,$$
(96)

$$\|\mathcal{R}_{\lambda}F\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})} \leq \|F\|_{L^{2}_{W\left(t,l_{\bar{\lambda}},m\right)}(\mathcal{I})} / |\Im\lambda|.$$

$$(97)$$

Let $F(t) \in L^{2}_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I}), G(t) \in L^{2}_{W(t,l_{\lambda},m)}(\mathcal{I})$ are \mathcal{H}^{r} -valued functions with compact support. We have

$$(\mathcal{R}_{\lambda}F,G)_{L^{2}_{W(t,l_{\lambda},m)}(\mathcal{I})} = (F,\mathcal{R}_{\bar{\lambda}},G)_{L^{2}_{W(t,l_{\bar{\lambda}},m)}(\mathcal{I})}.$$
(98)

since $M(\lambda) = M^*(\overline{\lambda})$. Due to inequalities (99), (100) equality (95) is valid for F(t), G(t) with non-compact support.

Now it follows from, (36), (101) that $\forall f(t), g(t) \in H$

$$m \left[R\left(\lambda\right) f, g \right] - m \left[f, R\left(\bar{\lambda}\right) g \right] = \left(\mathcal{R}_{\lambda} F\left(t, l_{\bar{\lambda}}, m\right), G\left(t, l_{\lambda}, m\right) \right)_{L^{2}_{W\left(t, l_{\lambda}, m\right)}\left(\mathcal{I}\right)} - \left(F\left(t, l_{\bar{\lambda}}, m\right), \mathcal{R}_{\bar{\lambda}} G\left(t, l_{\lambda}, m\right) \right)_{L^{2}_{W\left(t, l_{\bar{\lambda}}, m\right)}\left(\mathcal{I}\right)} = 0$$

Thus the closure of the operator $R(\lambda) f$ in $L^{2}_{m}(\mathcal{I})$ possesses property (90).

Since in view of (92) for any $f(t), g(t) \in H$

$$(R(\lambda) f, g)_{L^2_m(\alpha, \beta)} \to (R(\lambda) f, g)_{L^2_m(\mathcal{I})} \text{ as } (\alpha, \beta) \uparrow \mathcal{I}$$

uniformly in λ from any compact set from $\mathbb{C} \setminus \mathbb{R}^1$, we see that, in view of the analyticity of the operator function $M(\lambda)$ and vector-function $W(t, l_{\bar{\lambda}}, m) F(t, l_{\bar{\lambda}})$ (see (29) with $l = l_{\lambda}$) the operator $R(\lambda)$ depends analytically on the non-real λ in view of [19, p. 195]. Theorem 3.1 is proved.

For
$$r = 1$$
, $n_{\lambda}[y] = H_{\lambda}(t)y$ Theorem 3.1 is known [21].

Let us notice that if $L^2_m(\mathcal{I}) = \overset{\circ}{L^2_m}(\mathcal{I})$ then Theorem 3.1 is valid with $f(t) \in \overset{\circ}{H}$ instead of $f(t) \in H$ and without condition (77) with $P = I_r$ for infinite \mathcal{I} .

The following theorem establishes a relationship between the resolvents $R(\lambda)$ that are given by Theorem 3.1 and the boundary value problems for equation (1), (2) with boundary conditions depending on the spectral parameter. Similarly to the case $n_{\lambda}[y] \equiv 0$ [24] we see that the pair $\{y, f\}$ satisfies the boundary conditions that contain both y derivatives and f derivatives of corresponding orders at the ends of the interval.

Theorem 3.2. Let the interval $\mathcal{I} = (a, b)$ be finite and condition (77) with $P = I_r$ holds. Let the operator-functions $\mathcal{M}_{\lambda}, \mathcal{N}_{\lambda} \in B(\mathcal{H}^r)$ depend analytically on the non-real λ ,

$$\mathcal{M}_{\bar{\lambda}}^{*}\left[\Re Q\left(a,l_{\lambda}\right)\right]\mathcal{M}_{\lambda} = \mathcal{N}_{\bar{\lambda}}^{*}\left[\Re Q\left(b,l_{\lambda}\right)\right]\mathcal{N}_{\lambda} \quad \left(\Im\lambda \neq 0\right),\tag{99}$$

where $Q(t, l_{\lambda})$ is the coefficient of equation (76) corresponding by Theorem 1.1 to equation (1),

$$\|\mathcal{M}_{\lambda}h\| + \|\mathcal{N}_{\lambda}h\| > 0 \quad (0 \neq h \in \mathcal{H}^r, \Im\lambda \neq 0),$$
(100)

the lineal $\{\mathcal{M}_{\lambda}h \oplus \mathcal{N}_{\lambda}h | h \in \mathcal{H}^r\} \subset \mathcal{H}^{2r}$ is a maximal \mathcal{Q} -nonnegative subspace if $\Im \lambda \neq 0$, where $\mathcal{Q} = (\Im \lambda) \operatorname{diag} (\Re Q(a, l_{\lambda}), -\Re Q(b, l_{\lambda}))$ (and therefore

$$\Im\lambda\left(\mathcal{N}_{\lambda}^{*}\left[\Re Q\left(b,l_{\lambda}\right)\right]\mathcal{N}_{\lambda}-\mathcal{M}_{\lambda}^{*}\left[\Re Q\left(a,l_{\lambda}\right)\right]\mathcal{M}_{\lambda}\right)\leq0\quad\left(\Im\lambda\neq0\right)\right).$$
(101)

Then

1°. For any $f(t) \in H$ the boundary problem that is obtained by adding the boundary conditions

$$\exists h = h(\lambda, f) \in \mathcal{H}^r: \ \vec{y}(a, l_\lambda, m, f) = \mathcal{M}_\lambda h, \ \vec{y}(b, l_\lambda, m, f) = \mathcal{N}_\lambda h \tag{102}$$

to the equation (1), where $\vec{y}(t, l_{\lambda}, m, f)$ is defined by (28), has the unique solution $R(\lambda) f$ in $C^{r}(\bar{\mathcal{I}}, \mathcal{H})$ as $\Im \lambda \neq 0$. It is generated by the resolvent $R(\lambda)$ that is constructed, as in Theorem 3.1, using the characteristic operator

$$M(\lambda) = -\frac{1}{2} \left(X_{\lambda}^{-1}(a) \mathcal{M}_{\lambda} + X_{\lambda}^{-1}(b) \mathcal{N}_{\lambda} \right) \left(X_{\lambda}^{-1}(a) \mathcal{M}_{\lambda} - X_{\lambda}^{-1}(b) \mathcal{N}_{\lambda} \right)^{-1} (iG)^{-1}, \quad (103)$$

where

$$\left(X_{\lambda}^{-1}\left(a\right)\mathcal{M}_{\lambda}-X_{\lambda}^{-1}\left(b\right)\mathcal{N}_{\lambda}\right)^{-1}\in B\left(\mathcal{H}^{r}\right)\quad\left(\Im\lambda\neq0\right),$$

 $X_{\lambda}(t)$ is an operator solution of the homogeneous equation (76) such that $X_{\lambda}(0) = I_r$.

2°. For any operator $R(\lambda)$ from Theorem 3.1 vector-function $R(\lambda)f$ $(f \in H)$ is a solution of some boundary problem as in 1° .

Let us notice that if $f(t) \stackrel{H}{=} g(t)$ then in boundary conditions (102): $\vec{y}(t, l, m, f) = \vec{y}(t, l, m, g)$ in view of (28) and Proposition 3.1.

Proof. The proof of Theorem 3.2 follows from Theorems 1.1, 3.1 and from [22, Remark 1.1].

For the case $n_{\lambda}[y] \equiv 0$, Theorem 3.2 is known [22]. The example below show that the following is possible: for some resolvent $R(\lambda)$ from Theorem 3.1 $\exists f_0(t) \neq 0$ such that $m[f_0] = 0$ and therefore the "resolvent" equation (1) for $R(\lambda) f_0$ is homogeneous but $R(\lambda) f_0 \stackrel{H}{\neq} 0, \ \Im \lambda \neq 0.$

Example 3.2. Let m in (1) be such expression that equation m[f] = 0 has a solution $f_0(t) \neq 0$. Let in Theorem 3.2: $\mathcal{M}_{\lambda} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{N}_{\lambda} = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$, $R(\lambda)$ is the corresponding resolution for \mathcal{M}_{λ} . vent. Then $R(\lambda)f_0 \neq 0, \Im \lambda \neq 0$, while if $\mathcal{M}_{\lambda} = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \mathcal{N}_{\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}$ then for the corresponding resolvent $R(\lambda) f_0 \stackrel{H}{=} 0$, $\Im \lambda \neq 0$ (and therefore in view of [16, p. 87] $E_{\infty} f_0 = 0$ for generalized spectral family E_{μ} , which corresponds to $R(\lambda)$ by (3)).

It is known [16, p.86] that the operator-function $R(\lambda)$ (90)-(92) can be represented in the form

$$R(\lambda) = (T(\lambda) - \lambda)^{-1}, \qquad (104)$$

where $T(\lambda)$ is such linear relation that

$$\Im T(\lambda) \leq 0 \pmod{\pi}, \ T(\overline{\lambda}) = T^*(\lambda), \ \lambda \in \mathbb{C}^+,$$

the Cayley transform $C_{\mu}(T(\lambda))$ defines a holomorphic function in $\lambda \in \mathbb{C}_{+}$ for some (and hence for all) $\mu \in \mathbb{C}_+$. Applications of abstract relations of $T(\lambda)$ type (Nevanlinna families) to the theories of boundary relations and of generalized resolvents are proposed in [12, 13]

The description of $T(\lambda)$ corresponding to $R(\lambda)$ from Theorem 3.1 in regular case gives

Corollary 3.1. Let \mathcal{I} be finite and condition (77) with $P = I_r$ holds. Let us consider the relation $T(\lambda) = \overline{T'(\lambda)}$ as $\Im \lambda \neq 0$, where

$$T'(\lambda) = \left\{ \left\{ \tilde{y}(t), \tilde{f}(t) \right\} \middle| \tilde{y}(t) \stackrel{L^{2}_{m}(\mathcal{I})}{=} y(t) \in C^{r}(\bar{\mathcal{I}}), \tilde{f}(t) \stackrel{L^{2}_{m}(\mathcal{I})}{=} f(t) \in H, (l - n_{\lambda}) [y] = m[f], \\ \tilde{y}(t, l - n_{\lambda}, m, f) \text{ satisfy boundary condition} \\ \exists h = h(\lambda, f) \in \mathcal{H}^{r} : \tilde{y}(a, l - n_{\lambda}, m, f) = \mathcal{M}_{\lambda}h, \ \tilde{y}(b, l - n_{\lambda}, m, f) = \mathcal{N}_{\lambda}h, \\ \text{where operators } \mathcal{M}_{\lambda}, \mathcal{N}_{\lambda} \text{ satisfy the conditions of Theorem 3.2,} \\ \tilde{y}(t, l - n_{\lambda}, m, f) \stackrel{def}{=} \tilde{y}(t, l_{\lambda}, m, f) \mid_{m=0 \text{ in } l_{\lambda}} = \\ \left\{ \begin{pmatrix} \binom{n-1}{2} \oplus y^{(j)}(t) \\ \sum_{j=0}^{n} \oplus (y^{[s'-j]}(t) \mid l - n_{\lambda}) - f^{[s-j]}(t) \mid m) \end{pmatrix}, \qquad r = 2n \\ \begin{pmatrix} \binom{n-1}{2} \oplus y^{(j)}(t) \\ \sum_{j=1}^{n} \oplus (y^{[r-j]}(t) \mid l - n_{\lambda}) - f^{[s-j]}(t) \mid m) \end{pmatrix}, \qquad r = 2n + 1 > 1 \\ (here s' = order \ of \ expression \ l - n_{\lambda}, \\ y^{[0]}(t) \mid l - n_{\lambda}) = -\frac{i}{2} (q_{1}(t, \lambda) - \tilde{q}_{1}(t, \lambda)) y \ as \ s' = 1 \\ y^{[k']}(t) \mid l - n_{\lambda}) \equiv 0 \ as \ k' < \left[\frac{s'}{2}\right], \ f^{[k]}(t) \mid m \equiv 0 \ as \ k < \frac{s}{2} \end{pmatrix} \\ y(t), \qquad r = 1 \end{cases}$$
(105)

Then

1°. $(T(\lambda) - \lambda)^{-1}$ is equal to resolvent $R(\lambda)$ (88), (89) from Theorem 3.1 corresponding to characteristic operator $M(\lambda)$ (103).

2°. Let $R(\lambda)$ be resolvent (88), (89) from Theorem 3.1. Then $R(\lambda) = (T(\lambda) - \lambda)^{-1}$, where $T(\lambda)$ is some relation as in item 1°.

Proof. The proof follows from (28), Lemma 1.2, Theorem 3.2 and Remark 1.1 from [22]. \Box

Let in (1), (2) $n_{\lambda}[y] \equiv 0$ l.e. $l_{\lambda} = l - \lambda m$, where $l = l^*, m = m^*$ and coefficients of expressions m satisfy condition (74). We consider in $L_m^2(\mathcal{I})$ the linear relation

$$\mathcal{L}_{0}^{\prime} = \left\{ \left\{ \tilde{y}\left(t\right), \tilde{g}\left(t\right) \right\} | \tilde{y}\left(t\right) \stackrel{L_{m}^{2}\left(\mathcal{I}\right)}{=} y\left(t\right), \tilde{g}\left(t\right) \stackrel{L_{m}^{2}\left(\mathcal{I}\right)}{=} g\left(t\right), y\left(t\right) \in C^{r}\left(\bar{\mathcal{I}}, \mathcal{H}\right), g\left(t\right) \in H, l\left[y\right] = m\left[g\right], \\ \vec{y}\left(t, l, m, g\right) \text{ is equal to zero in the edge of } \mathcal{I} \text{ if this edge is finite and } \vec{y}\left(t, l, m, g\right)$$

is equal to zero in the some neighbourhood of the edge of \mathcal{I} if this edge is infinite $\left. \right\}$ (106)

³ where $\vec{y}(t, l, m, g)$ is defined by (105) with $l_{\lambda} = l - \lambda m$, f = g.

Below we assume that relation \mathcal{L}'_0 consists of the pairs of $\{y, g\}$ type.

The relation \mathcal{L}'_0 is symmetric due to the following Green formula with $\lambda_k = 0$:

³Let us notice that vector-function g(t) in (106) may be non-equal to zero in the finite edge or in the some neighbourhood of infinite edge of \mathcal{I} .

Let $y_k(t) \in C^r([\alpha, \beta], \mathcal{H}), f_k(t) \in C^s([\alpha, \beta], \mathcal{H}), \lambda_k \in \mathbb{C}, l[y_k] - \lambda_k m[y_k] = m[f_k], k = 1, 2$. Then

$$\int_{\alpha}^{\beta} m\{f_1, y_2\} dt - \int_{\alpha}^{\beta} m\{y_1, f_2\} dt + (\lambda_1 - \bar{\lambda}_2) \int_{\alpha}^{\beta} m\{y_1, y_2\} dt = i(\Re Q(t, l_{\lambda}) \vec{y}_1(t, l_{\lambda_1}, m, f_1), \vec{y}_2(t, l_{\lambda_2}, m, f_2))|_{\alpha}^{\beta}, \quad (107)$$

where $\vec{y}_k(t, l_{\lambda_k}, m, f_k)$ for $\lambda_k \in \mathbb{R}^1$ is defined by (105) with $l_{\lambda} = l - \lambda m$.

This formula is a corollary of Theorem 1.3 if $\Im \lambda_k \neq 0$. For its proof for example in the case $\lambda_1 \in \mathbb{R}^1$ we need to modify (42) for equation $l[y_1] - (\lambda_1 + i\varepsilon) m[y] = m[f_1 - i\varepsilon y_1]$ and then to pass to limit in (42) as $\varepsilon \to +0$.

In general the relation \mathcal{L}'_0 is not closed. We denote $\mathcal{L}_0 = \bar{\mathcal{L}}'_0$.

Theorem 3.3. Let $l_{\lambda} = l - \lambda m$ and the conditions of Theorem 3.1 hold. Then the operator $R(\lambda)$ from Theorem 3.1 is the generalized resolvent of the relation \mathcal{L}_0 . Let \mathcal{I} be finite and additionally the condition (77) hold. Then every such generalized resolvent can be constructed as the operator $R(\lambda)$.

Proof. In view of [16] and taking into account properties (90)-(92) of the operator $R(\lambda)$ it is sufficiently to prove that $R(\lambda) (\mathcal{L}_0 - \lambda) \subseteq \mathbf{I}$, where \mathbf{I} is a graph of the identical operator in $L^2_m(\mathcal{I})$. But this proposition is proved similarly to [22, p. 453] taking into account (107) and the fact that in view of (108) $\overrightarrow{(\tilde{y} - y)}(t, l - \lambda m, m, 0) = \overrightarrow{\tilde{y}}(t, l - \lambda m, m, g - \lambda y) - \overrightarrow{y}(t, l, m, g)$ if $\{y, g\} \in \mathcal{L}'_0, \ \widetilde{y} = R(\lambda)(g - \lambda y).$

Conversely let \mathcal{I} be finite, R_{λ} a generalized resolvent of relation \mathcal{L}_0 . We denote $N_{\lambda} = \{y(t) \in C^r(\mathcal{I}, \mathcal{H}), \lambda \in \mathcal{B}, l[y] - \lambda m[y] = 0\}$. We need the following

Lemma 3.1. Let condition (77) hold. Then the lineal N_{λ} is closed in $L^2_m(\mathcal{I})$.

Proof. The proof of Lemma 3.1 follows from (34).

Lemma 3.2. Let $\lambda \in \mathcal{B}$. Then $\overline{R\left(\mathcal{L}'_0 - \overline{\lambda}\right)} = N_{\lambda}^{\perp}$.

Proof. Let $x(t) \in N_{\lambda}$, $f(t) \in H$, y(t) is a solution of the following Cauchy problem:

$$l[y] - \bar{\lambda}m[y] = m[f], \ \vec{y}(a, l_{\bar{\lambda}}, m, f) = 0.$$
(108)

Then

$$m[f,x] = i(\Re Q(b,l_{\lambda})\vec{y}(b,l_{\bar{\lambda}},m,f),\vec{x}(b,l_{\lambda},m,0))$$
(109)

in view of Green formula (107). Therefore $\overline{R\left(\mathcal{L}'_0-\bar{\lambda}\right)} \subseteq N^{\perp}_{\lambda}$.

Let $g(t) \in N_{\lambda}^{\perp}$. Then $\exists H \ni g_n \xrightarrow{L_m^2(\mathcal{I})} g$, $g_n = x_n \oplus f_n$, $x_n \in N_{\lambda}$, $f_n \in N_{\lambda}^{\perp} \Rightarrow f_n \in H$. Let y_n be a solution of problem (108) with f_n instead of f. In view of (109) with $f = f_n$, one has: $\vec{y_n}(b, l_{\bar{\lambda}}, m, f_n) = 0 \Rightarrow f_n \in R\left(\mathcal{L}'_0 - \bar{\lambda}\right)$. But $f_n \xrightarrow{L_m^2(\mathcal{I})} g$. Therefore $\overline{R\left(\mathcal{L}'_0 - \bar{\lambda}\right)} \supseteq N_{\lambda}^{\perp}$. Lemma 3.2 is proved.

Lemma 3.3. Let the condition (77) hold, $\lambda \in \mathcal{B}$. Let $\left\{\tilde{y}, \tilde{f}\right\} \in \mathcal{L}_0^* - \lambda$, $\tilde{f} \stackrel{L^2_m(\mathcal{I})}{=} f \in H$. Then $\tilde{y} \stackrel{L^2_m(\mathcal{I})}{=} y \in C^r(\bar{\mathcal{I}}, \mathcal{H})$ and y(t) satisfies the equation (1).

Proof. Let $C^r(\bar{\mathcal{I}},\mathcal{H}) \ni y_0$ be a solution of (1). Let $\{\varphi,\psi\} \in \mathcal{L}'_0 - \bar{\lambda}$. Then $\vec{\varphi}(a, l_{\bar{\lambda}}, m, \psi) = \vec{\varphi}(b, l_{\bar{\lambda}}, m, \psi) = 0$ in view of (105), (106). Hence $m[\varphi, f] = m[\psi, y_0]$ due to Green formula (107). But $m[\varphi, f] = (\psi, \tilde{y})_{L^2_m(\mathcal{I})}$ in view of the definition of the adjoint relation. Hence

 $(\psi, \tilde{y} - y_0) = _{L^2_m(\mathcal{I})} 0.$ Therefore $\tilde{y} - y_0 \stackrel{L^2_m(\mathcal{I})}{=} y - y_0 \in N_\lambda$ in view of Lemmas 3.1, 3.2. Hence $\tilde{y} \stackrel{L^2_m(\mathcal{I})}{=} y \in C^r(\bar{\mathcal{I}}, \mathcal{H})$ and y is a solution of (1). Lemma 3.3 is proved.

We return to the proof of Theorem 3.3.

Let $f \in H$. Then in view of Lemma 3.3 $R_{\lambda} f \stackrel{L^2_m(\bar{\mathcal{I}})}{=} y \in C^r(\bar{\mathcal{I}}, \mathcal{H})$ and y satisfies equation (1). Therefore taking into account Theorem 1.1, [11, p.148] and (53) we have

$$y(t) = [X_{\lambda}(t)]_{1} \left\{ h - \frac{1}{2} (iG)^{-1} \left(\int_{a}^{b} sqn(s-t) X_{\bar{\lambda}}^{*}(s) W(s, l_{\bar{\lambda}}, m) F(s, l_{\bar{\lambda}}, m) ds \right) \right\}, \quad (110)$$

where $[X_{\lambda}(t)]_1 \in B(\mathcal{H}^r, \mathcal{H})$ is the first row of operator solution $X_{\lambda}(t)$ from Theorem 3.1, that is written in the matrix form, $h = h_{\lambda}(f) \in N^{\perp}$ is defined in the unique way in view of (34) and condition (77).

Let us prove that h depends on $I_{\lambda}f \stackrel{def}{=} \int_{a}^{b} X_{\overline{\lambda}}^{*}(s) W(s, l_{\overline{\lambda}}, m) F(s, l_{\overline{\lambda}}, m) ds$ in unique way. Operator $(I_{\lambda} : H \to N^{\perp})$ in view of Lemma 2.2. Moreover $I_{\lambda}N^{\perp} = N^{\perp}$ i.e. $\forall h_{0} \in N^{\perp} \exists f_{0} \in H : h_{0} = I_{\lambda}f_{0}$. For example we can set

$$f_0 = f_0(t,\lambda) = [X_{\bar{\lambda}}(t)]_1 \{ \Delta_{\bar{\lambda}}(\mathcal{I}) |_{N^{\perp}} \}^{-1} h_0$$
(111)

and to utilize the equality.

$$W(s, l_{\bar{\lambda}}, m) F_0(s, l_{\bar{\lambda}}, m) = W(s, l_{\bar{\lambda}}, m) X_{\bar{\lambda}}(s) \{\dots\}^{-1} h_0$$

If $f(t), g(t) \in H$ are such functions that $I_{\lambda}f = I_{\lambda}g$, then in view of (110)

$$\Im\lambda\left(\left(\Re Q\left(t,l_{\lambda}\right)\right)\overrightarrow{\Delta y}\left(t,l_{\lambda},m,f-g\right),\overrightarrow{\Delta y}\left(t,l_{\lambda},m,f-g\right)\right)\Big|_{\alpha}^{\beta} = \\ = \Im\lambda\left(\left(\Re Q\left(t,l_{\lambda}\right)\right)X_{\lambda}\left(t\right)\left(h_{\lambda}\left(f\right)-h_{\lambda}\left(g\right)\right),X_{\lambda}\left(t\right)\left(h_{\lambda}\left(f\right)-h_{\lambda}\left(g\right)\right)\right)\Big|_{\alpha}^{\beta}, \quad (112)$$

where $\Delta y = R_{\lambda} [f - g]$. But in view of (107) the left hand side of (112) is nonpositive since R_{λ} has property of (92) type. The right hand of (112) is nonnegative in view of (42). Hence $h_{\lambda} (f) = h_{\lambda} (g)$ in view of (42), (77). Thus h depends on $I_{\lambda} f$ in unique way and obviously in the linear way. Therefore

$$h = M\left(\lambda\right) I_{\lambda}f,\tag{113}$$

where $M(\lambda): N^{\perp} \to N^{\perp}$ is a linear operator and so $R_{\lambda}f$ $(f \in H)$ can be represented in the form (89).

Further, for definiteness, we will consider the most complicated case r = s = 2n.

Let us prove that $M(\lambda) \in B(N^{\perp})$, $\Im \lambda \neq 0$. Let $h_0 \in N^{\perp}$, $y = R_{\lambda} f_0$, where $f_0 = f_0(t, \lambda)$ see (111). Then in view of (110) and Theorem 1.1 we have

$$X_{\lambda}(t) M(\lambda) h_{0} = Y(t, l_{\lambda}, m) - \mathcal{F}_{0}(t, m) - \frac{1}{2} X_{\lambda}(t) (iG)^{-1} (I_{\lambda}(a, t) - I_{\lambda}(t, b)) F_{0}, \quad (114)$$

where $Y(t, l_{\lambda}, m)$, $F_0 = F_0(t, l_{\bar{\lambda}}, m)$, $\mathcal{F}_0(t, m)$ are defined by (26), (37) correspondingly with y and f_0 correspondingly instead of f, $I_{\lambda}(0, t) F_0$ is defined by (83). Therefore

$$\Delta_{\lambda}(a,b) M(\lambda)h_{0} = I_{\bar{\lambda}}y - I_{\bar{\lambda}}(a,b) \left(\mathcal{F}_{0}(t,m) + \frac{1}{2}X_{\lambda}(t)(iG)^{-1} \left(I_{\lambda}(a,t) - I_{\lambda}(t,b) \right) \mathcal{F}_{0} \right),$$
(115)

where $I_{\bar{\lambda}}y$, $I_{\bar{\lambda}}(a,b)(\ldots) \in N^{\perp}$ in view of (84). But

$$\forall g \in \mathcal{H}^r : |(I_{\bar{\lambda}}y,g)| \le \max_{t \in \bar{\mathcal{I}}} ||X_{\lambda}(t)|| \left\{ \int_{\mathcal{I}} ||W(t,l_{\lambda},m)|| dt \right\}^{1/2} ||R_{\lambda}f_0||_{L^2_m(\mathcal{I})} ||g||$$

in view of Cauchy inequality and (34). Therefore

$$\exists \text{ constant } c(\lambda): \ |(I_{\bar{\lambda}}y,g)| \le c(\lambda) \|y\| \|g\|$$
(116)

since

$$\|R_{\lambda}f_{0}\|_{L^{2}_{m}(\mathcal{I})} \leq \|\Delta_{\bar{\lambda}}(a,b)\|^{1/2} \left\| (\Delta_{\bar{\lambda}}(a,b)|_{N^{\perp}})^{-1} \right\| \|h_{0}\| / |\Im\lambda|$$

in view of (34), (116) and inequality: $||R_{\lambda}f_0||_{L^2_m(\mathcal{I})} \leq ||f_0||_{L^2_m(\mathcal{I})} / |\Im \lambda|$.

Obviously $|(I_{\bar{\lambda}}(a,b)(\ldots),g)|$ satisfies the estimate of type (116). Therefore $M(\lambda) \in B(N^{\perp})$.

Now we have to prove that $M(\lambda)$ is a characteristic operator o equation (73).

Let us prove that $M(\lambda)$ is strongly continuous for nonreal λ . To prove this fact it is enough to verify it for $\Delta_{\lambda}(a, b) M(\lambda)$; while the last one obviously follows from strongly continuity of vector-function $I_{\bar{\lambda}}R_{\lambda}f_0(t, \lambda)$.

In view of (34) we have $\forall g \in \mathcal{H}^r$

$$(I_{\bar{\lambda}}R_{\lambda}f_{0}(t,\lambda) - I_{\mu}R_{\bar{\mu}}f_{0}(t,\mu),g) = m [R_{\lambda}f_{0}(t,\lambda), [X_{\lambda}(t)]_{1}g] - m [R_{\mu}f_{0}(t,\mu), [X_{\mu}(t)]_{1}g].$$

Then the required statement can be derived from the equality

$$m\left\{ [X_{\lambda}(t) - X_{\mu}(t)]_{1} g, [X_{\lambda}(t) - X_{\mu}(t)]_{1} g \right\} =$$

$$= (W(t, l_{\lambda}, m) ((X_{\lambda}(t) - X_{\mu}(t)) g + (\lambda - \mu) \mathcal{F}(t, m)), (X_{\lambda}(t) - X_{\mu}(t)) g + (\lambda - \mu) \mathcal{F}(t, m)),$$
where $\mathcal{F}(t, m)$ is defined by (37) with $f(t) = [X_{\mu}(t)]_{1} g, ||X_{\lambda}(t) - X_{\mu}(t)|| \xrightarrow{\rightarrow} 0$ uniformly
in $t \in [a, b]$. and from the analogous equality for $m\{f_{0}(t, \lambda) - f_{0}(t, \mu), f_{0}(t, \lambda) - f_{0}(t, \mu)\}.$

Let us prove that $M(\lambda)$ is analytic for nonreal λ . To prove this fact it is enough in view of strongly continuity of $M(\lambda)$ to prove the analyticity in λ of $(I_{\lambda\mu}M(\lambda)I_{\lambda}f,g)$, where $f(t) \in C^r(\bar{\mathcal{I}},\mathcal{H}), g \in \mathcal{H}^r, (\Im\lambda)(\Im\mu) > 0$,

$$I_{\lambda\mu} = \int_{a}^{b} X_{\mu}^{*}(t) W(t, l_{\mu}, m) X_{\lambda}(t) dt \in B\left(N^{\perp}\right),$$

 $I_{\lambda\mu}^{-1} \in B(N^{\perp})$ if $|\lambda - \mu|$ is sufficiently small. In view of (115), (89), Theorem 1.1, (34), (29), (8) we have

$$(I_{\lambda\mu}M(\lambda) I_{\lambda}f,g) = m \left[R_{\lambda}f, \left[X_{\mu}(t) \right]_{1}g \right] + (\lambda - \mu) \int_{a}^{b} \left((R_{\lambda}f)^{[n]}(t \mid m), g^{(n)}(t) \right) dt + terms independent on R, f and analytic in \lambda$$

+ terms independent on $R_{\lambda}f$ and analytic in λ , (117)

where $g^{(n)}(t) \stackrel{def}{=} (p_n - \bar{\mu}\tilde{p}_n)^{-1} ([X_{\mu}]_1 g)^{[n]}(t \mid m)$. For scalar or vector-function $F(\lambda)$ let us denote

$$\Delta_{km}F\left(\lambda\right) = \frac{F\left(\lambda + \Delta_{k}\lambda\right) - F\left(\lambda\right)}{\Delta_{k}\lambda} - \frac{F\left(\lambda + \Delta_{m}\lambda\right) - F\left(\lambda\right)}{\Delta_{m}\lambda}.$$

Let us denote

$$\mathbf{R}_{n}\left(\lambda\right) = \int_{a}^{b} \left(\tilde{p}_{n}\left(R_{\lambda}f\right)^{[n]}\left(t|m\right), g^{(n)}\right) dt.$$

In view of (12), (74) we have

$$|\Delta_{km} \mathcal{R}_n(\lambda)| \le (m \left[\Delta_{km} \mathcal{R}_\lambda f, \, \Delta_{km} \mathcal{R}_\lambda f\right])^{\frac{1}{2}} \left(\int_a^b (\tilde{p}_n g^{(n)}, g^{(n)}) dt \right)^{1/2} \tag{118}$$

Therefore $R_n(\lambda)$ depends analytically on nonreal λ in view of analyticity of R_{λ} and so analyticity of $M(\lambda)$ is proved in view of (117).

Let us consider the solution $x_{\lambda}(t,F) = \mathcal{R}_{\lambda}F$ (79) of equation (73). Let us prove that $x_{\lambda}(t,F)$ satisfies the condition (55). Let us denote $y(t,\lambda,f) = R_{\lambda}f$. Then in view of Green formula (42)

$$m[y,y] - \frac{\Im m[y,f]}{\Im \lambda} = \frac{1}{2} \left(\Re Q(t,l_{\lambda}) \, \vec{y}(t,l_{\lambda},m,f) \,, \vec{y}(t,l_{\lambda},m,f) \right) \Big|_{a}^{b} / \Im \lambda \tag{119}$$

But the left hand side of (119) is ≤ 0 since $R_{\lambda}f$ is generalized resolvent. So

$$\forall f \in H: \ \Re\left(Q\left(t, l_{\lambda}\right) \vec{y}\left(t, l_{\lambda}, m, f\right), \vec{y}\left(t, l_{\lambda}, m, f\right)\right)|_{a}^{b} / \Im \lambda \leq 0.$$
(120)

But for every \mathcal{H}^{r} -valued $F(t) \in L^{2}_{W(t,l_{\bar{\lambda}},m)}(\bar{\mathcal{I}})$ there exists such vector-function $f(t) \in H$ that $x_{\lambda}(a,F) = \vec{y}(a,l_{\lambda},m,f), x_{\lambda}(b,F) = \vec{y}(b,l_{\lambda}m,f)$. So (55) is proved in view of (120).

To prove that $M(\lambda)$ is a characteristic operator of equation (73) it remains to show that $M(\lambda) = M^*(\lambda).$

Let us consider the following operator $\tilde{M}(\lambda) \in B(N^{\perp})$:

$$\tilde{M}(\lambda) = M(\lambda), \ \tilde{M}(\bar{\lambda}) = M^*(\lambda), \ \Im\lambda > 0$$

This operator is a characteristic operator of equation (73) in view of [22]. This characteristic operator generate by Theorem 3.1 the operator $R(\lambda)$ (89).

But $R(\lambda) = R_{\lambda}, \Im \lambda > 0 \Rightarrow R(\overline{\lambda}) = R^*(\lambda) = R^*_{\lambda} = R_{\overline{\lambda}}, \Im \lambda > 0 \Rightarrow \forall f \in H$:

$$\begin{aligned} \left\| \left[X_{\bar{\lambda}}\left(t\right) \right]_{1} \left(M^{*}\left(\lambda\right) - M\left(\bar{\lambda}\right) \right) \int_{a}^{b} X_{\lambda}^{*}\left(s\right) W\left(s, l_{\lambda}, m\right) F\left(s, l_{\lambda}, m\right) ds \right\|_{m} &= 0 \\ \Rightarrow \forall h \in N^{\perp} : \ \Delta_{\bar{\lambda}}\left(a, b\right) \left(M\left(\bar{\lambda}\right) - M^{*}\left(\lambda\right) \right) h = 0 \Rightarrow M\left(\bar{\lambda}\right) = M^{*}\left(\lambda\right). \end{aligned}$$
Theorem 3.3 is proved.

Theorem 3.3 is proved.

Let $\mathcal{I}_k, k = 1, 2$ be finite intervals, $\mathcal{I}_1 \subset \mathcal{I}_2$. Then, in spite of the fact that $f(t) \in C^s(\overline{\mathcal{I}}_2, \mathcal{H})$ but $\chi_{\mathcal{I}_1} f(t) \notin C^s(\overline{\mathcal{I}}_2, \mathcal{H})$, where $\chi_{\mathcal{I}_1}$ is the characteristic function of \mathcal{I}_1 , one has.

Corollary 3.2. Let $0 \in \mathcal{I}_1$ and the condition (77) with $\mathcal{I} = \mathcal{I}_2$ holds. Let R_{λ} be the generalized resolvent of the relation \mathcal{L}_0 in $L^2_m(\mathcal{I})$ with $\mathcal{I} = \mathcal{I}_2$. Then by Theorems 3.1, 3.3 there exists characteristic operator $M(\lambda)$ of equation (5) such that $R_{\lambda}f = y_1(t,\lambda,f)$ (88), $t \in \mathcal{I} = \mathcal{I}_2$, $f \in H (= H (\mathcal{I}_2))$. Let us define the operator $y_1^1 (t, \lambda, f) = R_\lambda^1 f, t \in \mathcal{I} = \mathcal{I}_1, f \in H (= H (\mathcal{I}_1))$ by the same formula (88) as operator $R_{\lambda}f$, but with $\mathcal{I} = \mathcal{I}_1$ instead of $\mathcal{I} = \mathcal{I}_2$. Then this operator is (after closing) the generalized resolvent of the relation \mathcal{L}_0 in $L^2_m(\mathcal{I})$ with $\mathcal{I} = \mathcal{I}_1$.

For generalized resolvents of differential operators a representation of (89) type was obtained in [37] for the scalar case and in [6] for the case of operator coefficients. For generalized resolvents for (1), (2) with s = 0, $n_{\lambda}[y] \equiv 0$ the representation of such a type was obtained in [7, 8, 20].

Therefore characteristic operator of equation (5) is an analogue of characteristic matrix from [37].

The resolvents of self-adjoint scalar differential operator in [17, p. 528], [30, p. 280] are represented in another form. Let us transform (89) to the form which is analogous to [17, p. 528], [30, p. 280].

Remark 3.1. Let us represent characteristic operator $M(\lambda)$ from Theorem 3.1 in the form (58). Then $R(\lambda)f$ (89) can be represented in the form

$$R(\lambda) f = \int_{a}^{t} \sum_{j=1}^{r} y_{j}(t,\lambda) \sum_{k=0}^{s/2} \left(x_{j}^{(k)}(s,\bar{\lambda}) \right)^{*} m_{k} [f(s)] ds + \int_{t}^{b} \sum_{j=1}^{r} x_{j}(t,\lambda) \sum_{k=0}^{s/2} \left(y_{j}^{(k)}(s,\bar{\lambda}) \right)^{*} m_{k} [f(s)] ds$$

where $x_j(t,\lambda), y_j(t,\lambda) \in B(\mathcal{H})$ are operator solutions of equation (1) as f = 0, such that $(x_1(t,\lambda), \ldots, x_r(t,\lambda))$ is the first row $[X_{\lambda}(t)]_1$ of operator matrix $X_{\lambda}(t), (y_1(t,\lambda), \ldots, y_r(t,\lambda)) = [X_{\lambda}(t)]_1 \mathcal{P}(\lambda) (iG)^{-1}, m_k [f(s)] = \tilde{p}_k (s) f^{(k)}(s) + \frac{i}{2} (\tilde{q}_k^*(s) f^{(k+1)}(s) - \tilde{q}_k (s) f^{(k-1)}(s)) (\tilde{q}_0 \equiv 0, \tilde{q}_{\frac{s}{2}+1} \equiv 0).$

Proof. In view of Theorem 1.2 one has

$$\forall h \in \mathcal{H}^{r} : \left(X_{\bar{\lambda}}^{*}(t) W_{\bar{\lambda}}(t) F_{\bar{\lambda}}(t), h\right) = m\left\{f(t), [X_{\bar{\lambda}}(t)]_{1}h\right\} = \\ = \left(\left(\left[X_{\lambda}(t)\right]_{1}^{*}, [X_{\lambda}(t)]_{1}^{'*}, \dots, [X_{\lambda}(t)]_{1}^{(n)^{*}}\right) col\left\{m_{0}\left[f(t)\right], m_{1}\left[f(t)\right], \dots, m_{s/2}\left[f(t)\right], 0, \dots, 0\right\}, h\right), \\ \left(n = \left[\frac{r}{2}\right]\right).$$

Now Remark 3.1 follows from (88)-(89) since $(\mathcal{P}(\lambda) - I_r)(iG)^{-1} = \left(\mathcal{P}(\bar{\lambda})(iG)^{-1}\right)^*$ in view of [22, p. 451].

Remark 3.1 shows that $(\mathcal{P}(\lambda) - I_r)(iG)^{-1}$ is an analogue of the matrix that is transponent to the matrix $\left\|\theta_{ij}^{-}(\lambda)\right\|$ from [17, p. 528] and is an analogue of characteristic matrix from [30, p. 280] $(\mathcal{P}(\lambda)(iG)^{-1})$ is an analogue of matrix that is transponent to the matrix $\left\|\theta_{ij}^{+}(\lambda)\right\|$ from [17, p. 528]).

If r is even, condition (55) is separated and a = c then formula (89) can be transformed to the form which is analogues to [30, p. 275-279].

Remark 3.2. Let r = 2n, a = c and condition (78) hold with $P = I_r$. Let for characteristic operator $M(\lambda)$ of equation (5) condition (55) be separated. (Therefore $M(\lambda)$ can be represented in the form (58) where characteristic projection $\mathcal{P}(\lambda)$ has the representation (64), (65), and equation (76) corresponding to equation (1) (f = 0) has a solutions $U_{\lambda}(t)$, $V_{\lambda}(t)$ (66)-(68)). Let domains D, D_1 are be the same as in Remark 2.1. Then for $\lambda \in D \bigcup D_1$ $R(\lambda)f$ (89) can be represented in the form

$$R(\lambda) f = \int_{a}^{t} \sum_{j=1}^{n} v_{j}(t,\lambda) \sum_{k=0}^{s/2} \left(u_{j}^{(k)}(s,\bar{\lambda}) \right)^{*} m_{k} [f(s)] ds + \int_{t}^{b} \sum_{j=1}^{n} u_{j}(t,\lambda) \sum_{k=0}^{s/2} \left(v_{j}^{(k)}(s,\bar{\lambda}) \right)^{*} m_{k} [f(s)] ds, \quad (121)$$

where $u_i(t,\lambda), v_i(t,\lambda) \in B(\mathcal{H})$ are operator solutions of equation (1) as f = 0, such that, $(u_1(t,\lambda),\ldots u_n(t,\lambda)) = [X_\lambda(t)]_1 \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix},$ $\left(v_{1}\left(t,\lambda\right),\ldots,v_{n}\left(t,\lambda\right)\right)=\left[X_{\lambda}\left(t\right)\right]_{1}\left(\begin{array}{c}b\left(\lambda\right)\\-a\left(\lambda\right)\end{array}\right)K^{-1}\left(\lambda\right)+\left(u_{1}\left(t,\lambda\right),\ldots,u_{n}\left(t,\lambda\right)\right)m_{a,b}\left(\lambda\right),$ (122)

 $K(\lambda), m_{a,b}(\lambda) \text{ see (67), (68); } (v_1(t,\lambda), \dots, v_n(t,\lambda)) h \in L^2_m(c,b) \forall h \in \mathcal{H}^n.$ Moreover if $\exists \lambda_0 \in \mathbb{C} \setminus \mathbb{R}^1$ such that $a(\lambda_0) = a(\overline{\lambda}_0)$, $b(\lambda_0) = b(\overline{\lambda}_0)$ then we can set $D = \mathbb{C}_+$ and

$$\left\|\left(v_{1}\left(t,\lambda\right),\ldots,v_{n}\left(t,\lambda\right)\right)h\right\|_{m}^{2} \leq \frac{\Im\left(m_{a,b}\left(\lambda\right)h,h\right)}{\Im\lambda} \left(\Im\lambda\neq0\right).$$

Proof. Proof of Remark 3.2 follows from Remark 2.1 and Theorem 1.2.

Remark 3.2 shows that operator-function $m_{a,b}(\lambda)$ from (121), (122) is an analogue of characteristic matrix from [30, p. 278] since for any self-adjoint operator initial condition (in particular for initial condition of [30, p. 277] type) the resolvent (121) exists such that solutionrow $(u_1(t,\lambda),\ldots,u_n(t,\lambda))$ satisfies this condition. For example if $a(\lambda) = I_n, b(\lambda) = b = b^*$ then $m_{I_n,b}(\lambda)$ is equal to characteristic matrix of [30, p. 276] type minus $b(I_n + b^2)^{-1}$.

Let us note that the connection between generalized resolvents of minimal operator corresponding to self-adjoint extension in Krein space and boundary value problem with boundary conditions depending on spectral parameter locally holomorphic in some set $\subset \mathbb{C} \setminus \mathbb{R}^1$ was studied in [15] for the scalar symmetric Sturm-Liouville operator on the semi-axis in limit point case.

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